BRIQ Newsletter Technical Manual[1]
DRAFT and INCOMPLETE

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[1]Selected portions of this document are based on Robert Brooks, *Building Quantitative Finance Applications with R*, forthcoming. Used with permission.

The BRIQ Newsletter Technical Manual derives from training materials for investment analysts on technical details of their vocation. It is provided here to support calculation performed in the production of the BRIQ Newsletter and related documents. Thus, way more technical details are provided than necessary, but hopefully it will prove useful for readers of the BRIQ Newsletter as well as those seeking to improve their understanding of quantitative financial analysis.

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Cautions and Disclaimers

The BRIQ Newsletter seeks to fill a need within the Biblically Responsible Investor (BRI) community. Specifically, the BRIQ Newsletter provides detailed analysis of various BRI funds. With any financial analysis, critical cautions and disclaimers are in order.

Critical cautions:

- Passive index fund comparators may be subject to debate and, at the end of the day, the sole discretion of us.
- Correctly measuring fund expenses is challenging, especially multi-class mutual funds. The analysis here seeks to be based on the lowest possible expense ratio found (Net analysis), such as, an institutional account. Some means of addressing the multitude of accounts for one mutual fund as well as differences between the net expense ratio and the gross expense ratio for a particular fund class will be addressed.
- The 1-, 3-, 5-year, and entire history (of what is available) performance alphas are currently not adjusted for survivorship bias likely skewing the aggregate performance numbers in favor of BRI funds. Some means of addressing this challenge may be developed in the future.

Disclaimers:

- Non-price information is primarily drawn from Fact Sheets made available at the beginning of the quarter where possible. For example, the BRIQ Newsletter Volume 24.4 non-price information is primarily drawn from Fact Sheets made available for September 30, 2024. When is information is not made on a timely basis by fund managers, a multitude of other sources are relied upon.
- There can be no assurance that the Fund's investment objectives or various categories of fund's investment objectives have been correctly identified.
- Investing involves risk, including the possible loss of principle. See James 4:13-17.
- Mutual funds (MFs) and exchange-traded funds (ETFs) are included within this analysis.
- Past performance is not indicative of future results and there can be no assurance that any reported past performance presented in this report will be achieved in the future.
- These materials do not constitute an offer to sell, or the solicitation of an offer to purchase, any MF or ETF.
- There is no assurance that the charges, risks, expenses and investment objectives reported here are accurate. There are based on our collection efforts that may be flawed.
- There are innumerable risks related to BRI-based investing. Please carefully perform your own analysis.
- Certain information contained in the BRIQ Newsletter, supporting technical documents, and work products may be deemed to contain "forward-looking statements." Due to various uncertainties, actual events or results or actual performance of the funds identified here as well as various categories of funds identified here may differ materially from those reported here.
- There may remain analytical errors and omissions. We seek to correct these issues as we are made aware of them and the ever present coding bugs are removed.

Technical Notes

- Price data is often not available the first few days of trading; hence, our dataset may start a day or two after closing prices begin to be reported.
- For older funds, the passive index fund may start after the BRI fund. In rare cases, the analysis starts with the oldest available passive index fund.
- When passive index fund is simply not available, the lowest fee active fund is used (e.g., market neutral funds).
- Cap and style based on Morningstar categorization, not fund categorization.
- Passive index funds are based on fund categorization with some attention paid to the resultant correlation.
- Although many BRI funds are based on some custom index, we do not categorize them as passive index funds unless the net expense ratio is less than 30 basis points.
- A detailed BRIQ Glossary is provided.
- Performance analysis is only on funds with complete data (did not start after the initial date).
- We seek to address survivorship bias in the future. The goal is to apply the set of funds available at the beginning of the period as the denominator and then establish the number of funds that are still in existence at the end of the period. The survivorship percentage will be the percentage of funds in existence at the beginning of the time period that are still in existence at the end of the time period.
- Arithmetic mean is compounded geometrically enabling comparison with geometric mean as arithmetic mean is known to be biased high. Simply multiplying corrupts this comparison.
- Aggregate performance is reported on a value-weighted basis, but could be reported on an equal-weighted basis.

About This Manual

This manual seeks to support investment analysts in their efforts to effectively and efficiently produce quality analytics that leads to improved decision making. Specifically, concrete guidance is provided here clarifying computational tasks associated with the appraisal of various investment instruments.

This manual is expected to be in continuous improvement and frequently supplemented with additional detail. Hence, particular attention should be given to the version number provided in the footer of each page.

This manual provides exact formulas for various calculations along with numerical examples for clarity. One use of this manual is for training purposes when new team members join your firm. Hence, many elementary details are provided. Further, given ambiguity within the investment profession, selected definitions are provided in the accompanying glossary.[2]

Although rarely identified and discussed, investment management as well as every other field of inquiry is based some presuppositions and assumptions. Investment analysts would do well to consider them carefully. Please see Appendix A for a brief introduction to presuppositions and assumptions made for much of the investment management vocation.

Instruments

This manual will focus on individual publicly traded stocks as well as stock portfolios. The term instrument is used generically to be either an individual stock, a portfolio of individual stocks, Exchange-Traded Funds (ETFs), and even mutual funds. Instruments could also be non-traded stock indexes or even measurement data, such as temperature. Thus, instrument is the most generic term.

Raw Data Sources

The source for raw data is fluid with some websites lasting longer than others. Investment analysts need to constantly monitor the integrity of their data sources regardless of fees paid to acquire it.

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[2] See www.briqnewsletter.com/???.

Interest rate data

The Board of Governors of the Federal Reserve System has published selected interest rate data for decades. The primary source for this data is found at the following website:

https://www.federalreserve.gov/releases/h15/

The Secured Overnight Funding Rate (SOFR) data can be found at the following website:

https://www.newyorkfed.org/markets/reference-rates/sofr-averages-and-index

Stock price data

There are numerous sources of stock price data from free to extremely expensive. It is vital to always remember that even though a high price is paid for cleansed data, it does not mean the data is correct. Investment analysts must continuously audit their raw data sources, particularly when experiencing stressed markets such as September 11, 2001.

Yahoo finance data

Selected financial data, including closing prices, are freely available at Yahoo finance:

https://finance.yahoo.com

There are many ways to access this type of data automatically with various computer programming languages. For example, in the language R, there is simple syntax to pull in this data.

Commodity Systems, Inc. (CSI)

CSI is a relatively inexpensive data source for stock and futures prices and related information. Once the different data files are downloaded, once can implement various computer programs.

Bloomberg

Bloomberg is a common data vendor but there are licensing issues related to pulling data from the Bloomberg platform and using it elsewhere. Please check with your compliance officer to be sure you are following Bloomberg's agreement.

Center for Research in Securities Prices (CRSP)

CRSP is a widely used data source for academics. The data tends to be stale but very robust and does not suffer from the problem of survivorship bias meaning to include only successful companies.

Other vendors

There are numerous other vendors and too many to list here. Investment analysts will always audit and validate periodically any data vendor as errors are way too common.

Return Calculations[3]

There are numerous ways to report return on an investment instrument. We document here several alternatives. At this point, we assume we have access to clean data and there are no unusual corporate actions, such as those addressed in the next section.

Calculating rates of return

As detailed in Appendix B, we assume the interim rate of return (*IRoR*) can be expressed as[4]

$$IRoR = \frac{EMV + Inc - BMV}{BMV}, \text{ (\textbf{Interim rate of return})} \tag{1}$$

where *EMV* denotes the ending market value, *Inc* denotes any transfer of consideration due to owning the position, and *BMV* denotes the beginning market value. Note that Equation (1) will be in decimal form and often rates of return are reported in percentage form, hence *IRoR* would be multiplied by 100 to get the percentage. Interim rates of return are linked consistent with GIPS.

Example 1. Interim rate of return

[3]See Appendix B. Return calculations.

[4]We assume discrete compounding here as it is the most common approach within the industry. For internal analysis, we also use the continuous compounding approach, or $\ln[(EMV + Inc)/BMV]$.

Suppose a stock is trading for \$15 per share on June 1st, on June 15th the stock pays a \$0.10 per share dividend and the stock closed on June 15th at \$17 per share.

The interim rate of return is simply

$$\begin{aligned} IRoR &= \frac{EMV + Inc - BMV}{BMV} \\ &= \frac{17.0 + 0.1 - 15.0}{15.0} = 0.14 \end{aligned}.$$

Thus, the 15 day rate of return is 14%.

Periodic rates of return

Let R_{dc} denote the discretely compounded periodic holding period return. Let V_t denote the financial instrument value observed at some initial time t ($V_t > 0$) and let $V_{t+\Delta t}$ denote the financial instrument value including all benefits received or costs paid based on owning the instrument over period Δt observed at some future time $t + \Delta t$ ($V_{t+\Delta t} \geq 0$). Thus, the discretely compounded periodic holding period return between time t and time $t + \Delta t$ is

$$R_{dc} = \frac{V_{t+\Delta t} - V_t}{V_t} = \frac{V_{t+\Delta t}}{V_t} - 1. \tag{2}$$

The periodic total return, a useful metric, represents the total value at the end of the period stemming from a one dollar investment at the beginning of the period or

$$TR_{\Delta t} = \frac{V_{t+\Delta t}}{V_t} = 1 + R_{dc}. \tag{3}$$

Rearranging, we can express the future value as

$$V_{t+\Delta t} = V_t\left(1 + R_{dc}\right) = V_t TR_{\Delta t}. \tag{4}$$

The continuously compounded periodic holding period return between time t and time $t+\Delta t$ is

$$R_{cc} \equiv \ln\left(\frac{V_{t+\Delta t}}{V_t}\right) = \ln\left(TR_{\Delta t}\right). \tag{5}$$

Rearranging, we can express the future value as

$$V_{t+\Delta t} = V_t e^{R_{cc}}. \tag{6}$$

Example 2. Periodic rate of return

Using the data in Example 1, we suppose a stock is trading for \$15 per share on June 1st, on June 15th the stock pays a \$0.10 per share dividend and the stock closed on June 15th at \$17 per share.

Thus, the initial value, $V_{t=6/1}$, is 15.0 and the terminal value, $V_{t+\Delta t=6/15}$, is 17.1. The discretely compounded periodic holding period return between time $t = 6/1$ and time $t + \Delta t = 6/15$ is

$$\begin{aligned} R_{dc} &= \frac{V_{t+\Delta t}}{V_t} - 1 \\ &= \frac{17.1}{15} - 1 = 0.14 \end{aligned}.$$

Thus, the interim rate of return found in Example 1 is equivalent to the periodic discretely compounded rate of return. The periodic total return is simply

$$\begin{aligned} TR_{\Delta t} &= 1 + R_{dc} \\ &= 1 + 0.14 = 1.14 \end{aligned}. \quad (7)$$

As a useful alternative, the continuously compounded periodic holding period return between 6/1 and 6/15 is

$$\begin{aligned} R_{cc} &= \ln\left(TR_{\Delta t}\right) \\ &= \ln\left(1.14\right) = 0.131028 \end{aligned} \text{ or } 13.1028\%. \quad (8)$$

Note that (with slight rounding error)

$$\begin{aligned} V_{6/15} &= V_{6/1} e^{R_{cc}} \\ &= 15.0 e^{0.131028} = 17.0 \end{aligned}. \quad (9)$$

Thus, the periodic rate of return can be expressed as 14% (discrete compounding), 1.14 (total return), or 13.1028 (continuous compounding).

Annalized discretely compounded holding period returns

Many statistics are reported in annualized terms. Unfortunately, there are several ways to annualize periodic returns ($R_{a,dc}$). Let m denote the compounding frequency per year. Thus, $m(\Delta t) = 1$. For example, monthly implies $m = 12$ and $\Delta t = 1/12$. Daily data raises a host of issues, such as how to treat week-ends, holidays, and leap years.

Geometric compounding of periodic discretely compounded holding period returns

The relationship between discretely compounded periodic holding period returns and geometrically annualized discretely compounded holding period returns can be expressed as

$$\left(1 + R_{dc}\right) = \left(1 + \frac{R_{a,dc}}{m}\right)^{m\Delta t}. \quad (10)$$

Thus, the geometrically annualized holding period returns can be expressed as

$$R_{a,dc} = m\left[\left(1 + R_{dc}\right)^{1/(m\Delta t)} - 1\right]. \quad (11)$$

Example 3. Geometric compounding of periodic discretely compounded holding period returns

Now suppose a stock is trading for \$50 per share on June 30th and two calendar months later August 31st the stock is trading for \$45. Assume no corporate actions, such as dividends or splits. We seek the annualized rate of return assuming monthly geometric compounding.

With this data, the bimonthly discrete compounded rate of return is

$$\begin{aligned} R_{dc} &= \frac{V_{t+\Delta t}}{V_t} - 1 \\ &= \frac{45.0}{50.0} - 1 = 0.9 - 1 = -0.10 \end{aligned} \text{ or } -10\%.$$

Thus, the annualized geometric compounding of monthly discretely compounded holding period return is ($m = 12$ for months and $\Delta t = 2/12$)

$$\begin{aligned} R_{a,dc} &= m\left[\left(1 + R_{dc}\right)^{1/(m\Delta t)} - 1\right] \\ &= 12\left[\left(1 + R_{dc}\right)^{1/\left(12\frac{2}{12}\right)} - 1\right] = -0.6158 \end{aligned} \text{ or } -61.58\%.$$

Note here the monthly rate of return is –5.13167 (=$[1 + (-0.1)]^{0.5} - 1$). Thus, a loss of 5.13167% monthly for 12 months results in a loss more than 61%.

Arithmetic compounding of periodic discretely compounded holding period returns

The relationship between discretely compounded periodic holding period returns and arithmetically annualized discretely compounded holding period returns can be expressed as

$$R_{dc} = \frac{R_{a,dc}}{m} = R_{a,dc}\left(\Delta t\right). \tag{12}$$

Thus, the arithmetically annualized holding period returns can be expressed as

$$R_{a,dc} = mR_{dc} = \frac{R_{dc}}{\Delta t}. \tag{13}$$

Example 4. Arithmetic compounding of periodic discretely compounded holding period returns

Using the data in Example 3, again a stock is trading for \$50 per share on June 30th and two calendar months later August 31st the stock is trading for \$45. Assume no corporate actions, such as dividends or splits. We now seek the annualized rate of return assuming *bimonthly* arithmetic compounding.

Recall we assume the bimonthly rate of return is based on discrete compounding, we found $R_{dc} = -10\%$. Thus, the annualized rate of return is simply

$$\begin{aligned} R_{a,dc} &= mR_{dc} \\ &= 6(-0.1) = -0.6 \end{aligned} \text{ or } -60\%. \tag{14}$$

Thus, we see the method of annualizing impact the reported annualized number.

Annualized continuously compounded holding period returns

With this notation, the relationship between the continuously compounded periodic holding period returns and annualized continuously compounded holding period returns is simply

$$\begin{aligned} R_{a,cc} &= \ln\left(\frac{V_{t+\Delta t}}{V_t}\right)\Big/\Delta t \\ &= \frac{R_{cc}}{\Delta t} = mR_{cc} \end{aligned}. \tag{15}$$

Thus, the annualized continuously compounded holding period returns can be expressed as

$$V_{t+\Delta t} = V_t e^{R_{a,cc}\Delta t}. \tag{16}$$

Example ?. Annualized continuously compounded holding period returns

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Cumulative annualized returns (CAR)

For various calculations describe later, we often need the cumulative annualized returns. Cumulative returns simply seek to determine the gains or losses from a one dollar investment in the instrument. The cumulative annualized return (*CAR*) converts the cumulative return into an annualized percentage or

$$CAR = \left(\prod_{i=1}^{n} TR_i\right)^{n(\Delta t)} - 1, \tag{17}$$

where n denotes the number of periods. Thus, two years of monthly data has $n = 24$ observations and $\Delta t = 1/12$.

Example ?. Title

Dollar returns or profit

Dollar profit or loss is useful especially when the initial financial instrument is zero or negative. The change in instrument value or dollar return is

$$\begin{aligned} \Delta V &\equiv Profit \\ &= V_{t+\Delta t} - V_t. \end{aligned} \tag{18}$$

Example ?. Title

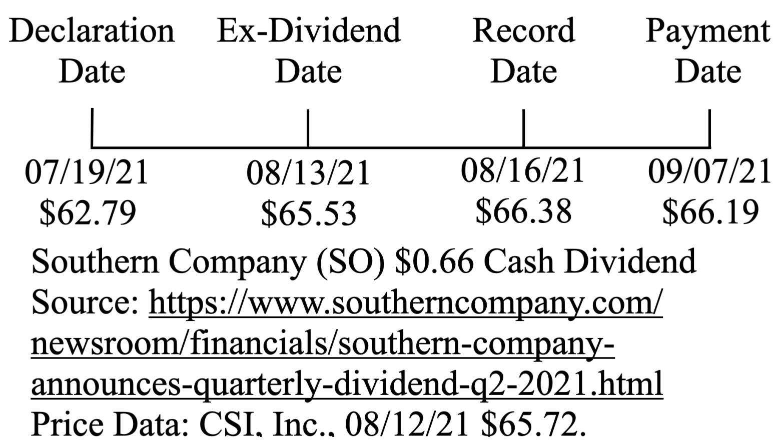
Known data management issues

Data management issues addressed here include dividends and missing data.

Dividend treatment

The payment of cash dividends on stocks is more complicated than it first appears. As Figure 1 illustrates, there are four important dates related to $0.66 per share Southern Company (SO) cash dividend. The data was taken from data vendor CSI, Inc. but it was consistent with Yahoo! Finance. Note that both data vendors backward adjust for dividends and other events.

Figure 1. Four Key Dates Related to one Southern Company Cash Dividend made in 2021



We now discuss each date:

Declaration date: This is the first day that the company announces the amount and timing of the next dividend. Any unique attributes regarding this dividend will be provided in this announcement. In this case, Southern Company released its dividend announcement on Wednesday, July 19, 2021.

Ex-dividend date: This is the first trading day when purchasing the share will not entitle you to receipt of this dividend. In this case, Southern Company's ex-dividend date is Friday, August 13, 2021. Thus, SO shares purchased on or before Thursday, August 12, 2021, would entitle the owner to receipt of this dividend, whereas SO shares purchased on or after the ex-dividend date would not entitle the owner to receipt of this dividend.

Record date: At this time, SO shares trade on the NYSE that settles T + 2, or two trading days after the transaction. Thus, owners of shares at the close of Thursday, August 12, 2021, will be holders on record T + 2 days later or Monday, August 16, 2021.

Payment date: Companies typically need several weeks to audit the holders of record to be sure who is and is not eligible to receive the dividend payments. For SO, the payment date is Tuesday, September 7, 2021.

Example ?. Title

Implied dividends

Many data services offer closing prices (P) and adjusted closing prices (A). Data services vary in how they compute adjusted closing prices. Based on the equivalence of total returns, we expect the following identity, subject to rounding error,

$$TR_P = \frac{P_t + D_t}{P_{t-1}} = \frac{A_t}{A_{t-1}} = TR_A. \tag{19}$$

Thus, we can solve for the implied dividend amount ($\hat{D}$) or

$$\hat{D}_t = \frac{A_t}{A_{t-1}} P_{t-1} - P_t. \tag{20}$$

In the SO example above, we note that on 08/12/21 the SO closing price was \$65.72 (the trading day before the ex-dividend date). Thus, the daily total return is

$$TR_P = \frac{P_t + D_t}{P_{t-1}} = \frac{65.53 + 0.66}{65.72} = 1.00715155,$$

and

$$TR_A = \frac{A_t}{A_{t-1}} = \frac{65.53}{65.06} = 1.00772241.$$

Alternatively, we can use as inputs the actual closing prices as well as adjusted closing prices to compute the implied dividend as

$$\hat{D}_t = \frac{A_t}{A_{t-1}} P_{t-1} - P_t = \frac{65.53}{65.06} 65.72 - 65.53 = 0.66476791.$$

Note the actual dividend was \$0.66, hence, we are (barely) within rounding error. One clear consequence of using adjusted closing prices is the potential cumulative effect of rounding error. It is always preferable to have access to the original raw data to routinely appraise the consequences of these types of errors.

One final audit would be to manually check the adjusted closing price as

$$A_{t-1} = \frac{A_t}{P_t + D_t} P_{t-1} = \frac{65.53}{65.53 + 0.66} 65.72 = 65.0646865.$$

Again, note that we confirm (barely) the reported rounding error at second decimal is 65.06.

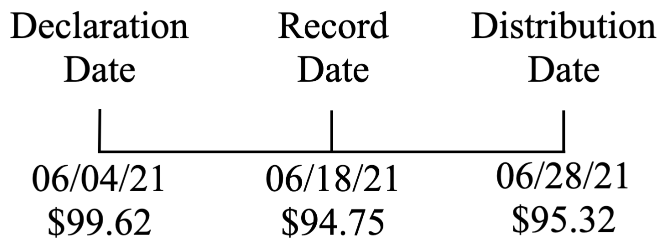
Example ?. Title

Splits

From time to time, managers will announce a stock split. For example, on June 4, 2021, CSX, Inc. announced a three-for-one stock split. Note that the additional company issuance of new shares with a split does not dilute shareholders' ownership interests. If a company simply issued new shares in the market, the existing ownership interest is diluted.

Like cash dividends, there are several important dates. Unfortunately, unlike cash dividends, the terminology changes. As Figure 2 illustrates, there are three important dates related to CSX, Inc.' 3-for-1 stock split announced in 2021. The stock price data was taken from data vendor CSI, Inc. Unfortunately, Yahoo! Finance adjusts all stock prices for splits, so it is not possible to observe pre-split prices.

Figure 2. Three Key Dates Related to CSX Stock Split made in 2021



CSX, Inc. (CSX) 3-for-1 stock split.
Source: https://investors.csx.com/news-and-events/news/news-details/2021/CSX-Announces-Stock-Split/default.aspx
Price Data: CSI, Inc.
06/21/21 $95.18 (Ex-record date)
06/29/21 $31.56 (Ex-distribution date)

We now discuss each date:

Declaration date: This is the first day that the company announces the stock split terms and related dates. Any unique attributes regarding this stock split will be provided in this announcement. In this case, CSX released its stock split announcement on Friday, June 4, 2021.

Record date: Unlike cash dividends, a stock split record date comes prior to any ex-date. Based on the announcement, owners of shares at the close of Friday, June 18, 2021, will be entitled to receive two additional shares of stock on Monday, June 28, 2021.

Ex-record date: This is the first trading day when purchasing the share will not entitle you to receipt of additional shares from the company. In this case, CSX's ex-split date is Monday, June 21, 2021. Again unlike cash dividends, purchase of CSX's shares between and including the ex-record date and the distribution date does entitle the purchaser to receive two additional shares from the seller on the distribution date.

Distribution date: Companies typically need a period of time to audit the holders of record to be sure who is and is not eligible to receive the additional shares. For CSX, the distribution date is Monday, June 28, 2021. After the close of trading on the distribution date, shareholders on the record date receive two additional shares.

Ex-distribution date: The next trading day after the distribution date when shares trade on a fully diluted basis. For CSX, the ex-distribution date is Tuesday, June 29, 2021.

Note that reverse stock splits work in the same way except the number of outstanding shares is reduced. Hence, with a 1-for-3 reverse stock split, each existing shareholder would see their number of shares reduced by one third.

Example ?. Title

Spinoffs and other corporate actions
Companies engage in a myriad of unique actions from mergers, spinoffs, and even in-kind dividends (e.g., ski lift tickets in lieu of cash dividends). In each unique case, great care must be given to treating these actions consistently to create a legitimate representation of investor performance.

For example, American Outdoor Brands Corporation executed a spinoff in 2020. With many corporate actions, the formal announcement date is often difficult to ascertain. For example, on November 19, 2019, American Outdoor Brands Corporation (AOBC) announced its intention to create two independent publicly traded companies (Smith & Wesson Brands, Inc. (SWBI) and American Outdoor Brands, Inc. (AOUT)). In the November 19, 2019, announcement the spinoff "… is expected to be completed in the second half of calendar 2020, would create two independent publicly traded companies: Smith & Wesson Brands, Inc. (which would encompass the firearm business) and American Outdoor Brands, Inc. (which would encompass the outdoor products and accessories business." There were no specific details in this announcement.

On July 31, 2020, the company announced the specific terms of the spinoff. Interestingly, AOBC first rebranded to SWBI and subsequently spun off AOUT. All transactions were structured to be tax-free. In this case, the record date is August 10, 2020, and the distribution date is August 24, 2020. For every 4 shares of SWBI owned at the close of August 10, 2020, the shareholders will receive 1 share of AOUT.

Thus, accurately computing the appropriate individual instrument's rate of return requires several decisions. We need to determine how to adjust SWBI's rate of return due to the spinoff. Further, we may need to pick up the rate of return series of AOUT should the portfolio retain it after the spinoff.

Based on the data available, it appears that selling SWBI between and including August 11, 2020, through August 24, 2020, would obligate delivery of one share of AOUT for every four shares sold. Beginning on August 25, 2020, SWBI traded independently of AOUT. The available closing price data were as follows:

Date\Ticker	SWBI	AOUT
August 24, 2020	$20.91	$19.40
August 25, 2020	$17.27	$17.78

Based on this data, the daily discretely compounded return for SWBI is[5]

$$\begin{aligned}R_{SWBI} &= \frac{P_{SWBI,8/25} + P_{AOUT,8/25}/4}{P_{SWBI,8/24}} - 1 \\ &= \frac{17.27 + 17.78/4}{20.91} - 1 \\ &= 0.0385\end{aligned}$$

or 3.85%. Thus, the spinoff AOUT is assumed sold and immediately reinvested in SWBI.

Example ?. Title

Managed portfolios
Dividends and other corporate actions of managed portfolios, such as ETFs, can get complicated. A cash dividend made by an ETF is assumed to be reinvested in that ETF. For legitimate return calculations, it is assumed that the cash dividend is used to purchase more shares of the ETF.

Missing data
Missing data occurs for a host of reasons. We identify just a few here. Our focus is data based on daily closing prices.

[5] Rate of return calculations are covered in detail later.

Weekends
For most countries, weekends are Saturday and Sunday, and active markets are closed. Increasingly, there are 24/7 (24 hours a day, 7 days per week) trading opportunities but often the bid-asked spreads are very large. Again, we focus only on daily closing prices and related data, such as volume, daily high price, daily low price, and daily open price.

The treatment of missing data on Saturday and Sunday has some interesting challenges, especially within options markets where time to maturity is critical. Most raw daily datasets contain only data from weekdays, Monday through Friday. If only weekdays are counted, then the total number of days in the year is around 260 (= 52 weeks x 5 days/week).

There are some challenges when the holding period is assumed to be monthly as months start and end on all days of the week, not just weekdays. Typically, the analysis required needs to specify whether the monthly measurement is last trading day, first trading day, or some other point in the month. We a complete set of weekday data, other periodicities can be handled. The key is routine holidays and other events must be addressed for the purpose of generating a weekday dataset with no missing data.

Example ?. Title

Routine holidays
Data is often missing from various data sets for a host of reasons. For example, when working with weekday closing prices, there are many holidays when markets are closed. These closures occur primarily on Monday and Fridays, but can occur on other weekdays, such as the fourth Thursday in November currently for Thanksgiving.

For routine holiday closures, we apply the forward fill procedure. Specifically, the previous trading day's closing price is simply copied forward, and the holiday has a closing price. One consequence of the forward fill procedure is having too many zero rates of return than would be expected under any assumed distribution. This is not a serious problem as all the statistical measures will have the same bias. For example, correlations will be slightly higher since all instruments moved in the same way (zero).

Example ?. Title

Short-term market disruptions
Due to natural disasters or man-made actions, there are times when financial markets or some components of financial markets are shut down. For example, the New York Stock Exchange closed on Tuesday, September 11, 2001, following the Islamist terrorist attack on the U.S. by the radical group al-Qaeda. The NYSE reopened on Monday, September 17, 2001. Thus, closing prices were absent for four days. Following several data vendors, such as Commodity Systems, Inc., forward fill procedure will also be used.

Given the unusual nature of these types of disruptions, professional judgment will be exercised by the analyst when deciding on the appropriate procedure to deploy. One key issue is clear documentation to future analysts will know how it was handled.

Example ?. Title

Long-term missing data series
When significant time series data is missing, an effort will be made to find suitable proxies. For example, suppose we needed a proxy for the S&P 500 Real Estate sector ETF (XLRE) between 12/22/1998 when other sector ETFs began trading and 10/08/2015, the initiation of XLRE trading. Obviously, forward fill with zero

return is inadequate for this extended period. In this case, a suitable proxy would be sought that best represents the stochastic behavior of XLRE.

Example ?. Title

Individual instrument statistics

Individual instrument statistics can be used to appraise various portfolio, such as ETFs or SMAs, as well as appraise individual stocks held within these portfolios. One important issue is understanding various anticipated co-movements between various products. Note that the use of expectations here assumes some objectively given distribution, such as the distribution derived from historical data.

Mean

Sample arithmetic mean is also often called the mean or average. The population mean is the first moment of the distribution, μ_1.

$\mu_1 = E(R)$ and (**First moment, population mean, or population average**) (21)

$$\bar{R}_A = \frac{1}{n}\sum_{i=1}^{n} R_i. \text{ (Sample average)} \tag{22}$$

Example ?. Arithmetic average

Consider the canonical example of a stock at time 0 is trading for \$100, then at time 1 is trading for \$200, and finally at time 2 is trading back at \$100. Clearly, the investor who bought at time 0 and sold at time 2 had no profit or loss.

In the case, R_1 = 100% [= (200-100)/100 = 1] and R_2 = –50% [= (100 – 200)/200]. Thus, the arithmetic average is

$$\begin{aligned} \bar{R}_A &= \frac{1}{n}\sum_{i=1}^{n} R_i \\ &= \frac{1}{2}(1.0 - 0.5) = 0.25 \end{aligned},$$

or 25%. Clearly, the arithmetic average is biased high from an intuitive investment return perspective. There are times, however, that an investment analyst will need an arithmetic average.

Median

The median is halfway through an ordered set of data or the probability density function. That is,

$\hat{x}$ such that $\int_0^{\hat{x}} f(x)dx = \int_{\hat{x}}^{+\infty} f(x)dx = \frac{1}{2}$.

Mode

The most frequently observed value within a data set or the peak of the probability density function. The mode satisfies the following two properties, $f'(x) = 0$ and $f''(x) < 0$.

Cumulative returns

The cumulative return is the difference between the terminal value and an initial \$1 investment. Thus, the cumulative return (CR) can be expressed as

$$CR = \prod_{i=1}^{n} TR_i - 1. \text{ (Cumulative Return)} \tag{23}$$

Example ?. Cumulative returns

Consider again a stock at time 0 is trading for \$100, then at time 1 (1 year later) is trading for \$200, and finally at time 2 (2 years later) is trading back at \$100. The investor who bought at time 0 and sold at time 2 had no profit or loss. Again, R_1 = 100% and R_2 = –50%. Based on our definition of total return, we have

$$TR_1 = \frac{P_1}{P_0} = \frac{200}{100} = 2.0 \text{ and } TR_2 = \frac{P_2}{P_1} = \frac{100}{200} = 0.5.$$

Thus, the cumulative return (CR) is

$$CR = \prod_{i=1}^{n} TR_i - 1 = 2(0.5) - 1 = 0.$$

Thus, the cumulative return is zero supporting intuition since we started with \$100 and ended with \$100.

The cumulative annualized return (CAR)

$$CAR = \left(\prod_{i=1}^{n} TR_i\right)^{1/[n(\Delta t)]} - 1, \tag{24}$$

where n denotes the number of total returns in the series and Δt denotes the fraction of the year in each period.

Example ?. Cumulative annualized returns

Consider again a stock at time 0 is trading for \$100, then at time 1 (1 year later) is trading for \$200, and finally at time 2 (2 years later) is trading back at \$100. The investor who bought at time 0 and sold at time 2 had no profit or loss. Again, R_1 = 100% and R_2 = –50%. Based on our definition of total return, we have

$$TR_1 = \frac{P_1}{P_0} = \frac{200}{100} = 2.0 \text{ and } TR_2 = \frac{P_2}{P_1} = \frac{100}{200} = 0.5.$$

Thus, the cumulative annualized return (CAR) is

$$CAR = \left(\prod_{i=1}^{n} TR_i\right)^{1/[n(\Delta t)]} - 1 = 2(0.5) - 1 = 0.$$

Thus, the cumulative return is zero supporting intuition since we started with \$100 and ended with \$100.

Geometric average is based on the product ex-post rates of return over the time periods and then takes the n^{th} root to express the average on a per period basis. There are several different ways to express the geometric average as illustrated below.

$$\bar{R}_G = \left[\prod_{i=1}^{n}(1+R_i)\right]^{\frac{1}{n}} - 1 = \left[\prod_{i=1}^{n} TR_i\right]^{\frac{1}{n}} - 1 = (1+CR)^{1/n} - 1. \textbf{ (Geometric average)} \tag{25}$$

Example ?. Geometric average

Again, consider the previous example of a stock at time 0 is trading for \$100, then at time 1 is trading for \$200, and finally at time 2 is trading back at \$100. Clearly, the investor who bought at time 0 and sold at time 2 had no profit or loss.

Again, R_1 = 100% and R_2 = –50%. Thus, the geometric average is

$$\bar{R}_G = \left[\prod_{i=1}^{n}(1+R_i)\right]^{\frac{1}{n}} - 1 \\ = \left[(1+1)(1-0.5)\right]^{\frac{1}{2}} - 1 = 0,$$

or 0%. The geometric average is yielding the intuitive investment return.

When severe outliers are a problem, the harmonic mean can be used. The harmonic average is often used with financial multiples like market-to-book ratio and price-to-earnings ratio. For illustration purposes only, we can compute the harmonic average of total return and then subtract one.

$$\bar{R}_H = n\left[\sum_{i=1}^{n}\frac{1}{(1+R_i)}\right]^{-1} - 1. \text{ (\textbf{Harmonic average})} \tag{26}$$

Note that the arithmetic mean is greater than the geometric mean. The geometric mean is also greater than the harmonic mean.

Example ?. Harmonic average

The harmonic average of five P/E ratios, 10/1, 50/2, 40/0, 70/5, and 30/4, is

$$P\bar{E}_H = n\left(\sum_{i=1}^{n}\frac{1}{P/E}\right)^{-1} = 5\left(\frac{1}{10}+\frac{2}{50}+\frac{0}{40}+\frac{5}{70}+\frac{4}{30}\right)^{-1} = 5(0.1+0.04+0+0.07143+0.13333)^{-1} \\ = 5(0.34476)^{-1} = 14.5.$$

Note that the arithmetic average of P/E is undefined because of the third observation, 40/0. Alternatively, we can compute the arithmetic average of E/P and then take its inverse to compute the harmonic mean or

$$E\bar{P}_A = \frac{1}{n}\sum_{i=1}^{n}\frac{E}{P} = \frac{1}{5}\left(\frac{1}{10}+\frac{2}{50}+\frac{0}{40}+\frac{5}{70}+\frac{4}{30}\right) = 0.2(0.1+0.04+0+0.07143+0.13333) \\ = 0.2(0.34476) = 0.068952.$$

Thus, the inverse is 14.5 (= 1/0.068952).

Example ?. Harmonic average

Again consider the previous example of a stock at time 0 is trading for \$100, then at time 1 is trading for \$200, and finally at time 2 is trading back at \$100.

Again, R_1 = 100% and R_2 = –50%. Thus, the harmonic average is

$$\bar{R}_H = n\left[\sum_{i=1}^{n}\frac{1}{(1+R_i)}\right]^{-1} - 1 \\ = 2\left[\frac{1}{(1+1)}+\frac{1}{(1-0.5)}\right]^{-1} - 1 = \frac{2}{\frac{1}{2}+\frac{1}{0.5}} - 1, \\ = \frac{2}{2.5} - 1 = -0.2$$

or –20%. The harmonic average is not yielding any sort of intuitive investment return. It is, however, useful for averaging ratios that may contain zero in the denominator but not the numerator.

Standard deviation

The standard deviation is a measure of dispersion and is based on the variance. The variance is based on the square of deviations from the mean. Thus, the units of measure for variance is in the underlying units (e.g., dollars or percent) squared. By taking the standard deviation of the variance, we have a dispersion measure in the same units has the underlying observations.

The population variance is the second moment about the mean, μ_2.

$$\begin{aligned}\mu_2 &= \sigma_p^2 \\ &= E\left(R-\bar{R}_A\right)^2\end{aligned}. \textbf{ (Population variance)} \tag{27}$$

With the entire discrete population, the population variance can be expressed as

$$\begin{aligned}\mu_2 &= \frac{1}{n}\sum_{i=1}^{n}\left(R_i-\bar{R}_A\right)^2 \\ &= \frac{1}{n}\sum_{i=1}^{n}R_i^2 - \frac{2}{n}\bar{R}_A\sum_{i=1}^{n}R_i + \frac{n}{n}\bar{R}_A^2 \\ &= \frac{1}{n}\sum_{i=1}^{n}R_i^2 - \frac{2}{n}\bar{R}_A n\bar{R}_A + \bar{R}_A^2 \\ &= \frac{1}{n}\sum_{i=1}^{n}R_i^2 - \bar{R}_A^2\end{aligned}. \tag{28}$$

With limited sample data, the unbiased sample variance can be expressed as

$$\begin{aligned}\sigma_s^2 &= \frac{n}{n-1}\sigma_p^2 \\ &= \frac{n}{n-1}\left(\frac{1}{n}\sum_{i=1}^{n}R_i^2 - \bar{R}_A^2\right) \\ &= \frac{1}{n-1}\sum_{i=1}^{n}R_i^2 - \frac{n}{n-1}\bar{R}_A^2\end{aligned}. \textbf{ (Sample variance)} \tag{29}$$

The square root of the variance, known as standard deviation, can therefore be expressed as:

$$\begin{aligned}\sigma_p &= \left[E\left(R-\bar{R}_A\right)^2\right]^{1/2} \\ &= \left(\frac{1}{n}\sum_{i=1}^{n}R_i^2 - \bar{R}_A^2\right)^{1/2}\end{aligned}. \textbf{ (Population standard deviation)} \tag{30}$$

$$\begin{aligned}\sigma_s &= \left[\frac{1}{n-1}\sum_{i=1}^{n}\left(R_i-\bar{R}_A\right)^2\right]^{1/2} \\ &= \left(\frac{1}{n-1}\sum_{i=1}^{n}R_i^2 - \frac{n}{n-1}\bar{R}_A^2\right)^{1/2}\end{aligned}. \textbf{ (Sample standard deviation)} \tag{31}$$

Example ?. Title

Skewness

Skewness is the third standardized moment and measures asymmetry in the distribution of the sample. Symmetric distributions will have a population skewness of zero and a sample skewness near zero. Negative values for this statistic indicate that the distribution is skewed left, and the left tail of the distribution is longer. Positive values for this statistic indicate that the distribution is skewed right, and the right tail of the distribution is longer.

The third moment about the mean, μ_3, can be expressed as

$$\mu_3 = E\left(R - \bar{R}_A\right)^3. \text{ (\textbf{Third moment about the mean})} \tag{32}$$

With the entire discrete population, the population third moment about the mean can be expressed as

$$\begin{aligned}\mu_3 &= \frac{1}{n}\sum_{i=1}^{n}\left(R_i - \bar{R}_A\right)^3 \\ &= \frac{1}{n}\sum_{i=1}^{n}R_i^3 - \frac{3}{n}\bar{R}_A\sum_{i=1}^{n}R_i^2 + \frac{3}{n}\bar{R}_A^2\sum_{i=1}^{n}R_i + \frac{n}{n}\bar{R}_A^3. \\ &= \frac{1}{n}\sum_{i=1}^{n}R_i^3 - \frac{3\bar{R}_A}{n}\sum_{i=1}^{n}R_i^2 - (n-3)\bar{R}_A^3\end{aligned} \tag{33}$$

Population skewness can be expressed as

$$\gamma_{S,p} = \frac{\mu_3}{\sigma_p^3}. \text{ (\textbf{Population skewness})} \tag{34}$$

With limited sample data, the unbiased sample skewness can be expressed as

$$\begin{aligned}\gamma_{S,s} &= \frac{\sqrt{n(n-1)}}{n-2}\frac{\mu_3}{\sigma_p^3} \\ &= \frac{\sqrt{n(n-1)}}{n-2}\frac{\frac{1}{n}\sum_{i=1}^{n}\left(R_i - \bar{R}_A\right)^3}{\left[\frac{1}{n}\sum_{i=1}^{n}\left(R_i - \bar{R}_A\right)^2\right]^{3/2}} \\ &= \frac{\sqrt{n(n-1)}}{n-2}\frac{\frac{1}{n}\sum_{i=1}^{n}R_i^3 - \frac{3\bar{R}_A}{n}\sum_{i=1}^{n}R_i^2 - (n-3)\bar{R}_A^3}{\left(\frac{1}{n}\sum_{i=1}^{n}R_i^2 - \bar{R}_A^2\right)^{3/2}}\end{aligned}. \text{ (\textbf{Sample skewness})} \tag{35}$$

If skewness is positive, then typically mean > median > mode. If skewness is negative, then typically mean < median < mode. See Stuart and Ord (1987), p. 107.

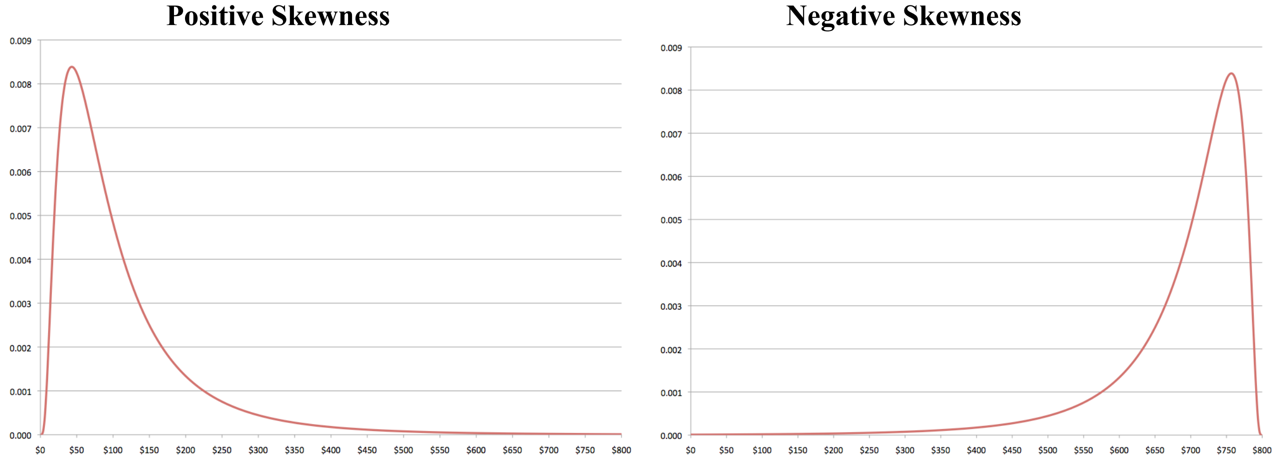
Some have categorized skewness in the following manner:

- Highly skewed: $-1 < \mu_3$ or $\mu_3 > +1$
- Moderately skewed: $-1 < \mu_3 < -1/2$ or $1/2 > \mu_3 > +1$
- Approximately symmetric: $-1/2 < \mu_3 < 1/2$

Figure 3 illustrates positive and negative skewness. The positive skewness is a lognormal distribution (stock price = $100, expected return = 12%, and standard deviation = 30%). The negative skewness is a shifted lognormal distribution of $800 minus the previous distribution.

<< discuss investor preference for less negative or more positive skewness >>

Figure 3. Illustration of Positive and Negative Skewness



Example ?. Title

Kurtosis

Kurtosis is the fourth standardized moment and measures the height and sharpness of the central peak of the distribution relative to the normal distribution. Higher kurtosis implies higher and sharper central peak whereas lower kurtosis implies lower and flatter central peak. The normal distribution has a kurtosis of 3. Therefore, it is customary to subtract three from kurtosis and report what is termed excess kurtosis. Often excess kurtosis is just referred to as kurtosis. One must always know how to interpret the reported kurtosis. That is, whether it has 3 subtracted or not.

The fourth moment about the mean, μ_4, can be expressed as

$$\mu_4 = E\left(R - \bar{R}_A\right)^4. \textbf{ (Fourth moment about the mean)} \tag{36}$$

With the entire discrete population, the population fourth moment about the mean can be expressed as

$$\begin{aligned} \mu_4 &= \frac{1}{n}\sum_{i=1}^{n}\left(R_i - \bar{R}_A\right)^4 \\ &= \frac{1}{n}\sum_{i=1}^{n}R_i^4 - \frac{4\bar{R}_A}{n}\sum_{i=1}^{n}R_i^3 + \frac{6\bar{R}_A^2}{n}\sum_{i=1}^{n}R_i^2 - \frac{4\bar{R}_A^3}{n}\sum_{i=1}^{n}R_i + \frac{n}{n}\bar{R}_A^4. \\ &= \frac{1}{n}\sum_{i=1}^{n}R_i^4 - \frac{4\bar{R}_A}{n}\sum_{i=1}^{n}R_i^2 + \frac{6\bar{R}_A^2}{n}\sum_{i=1}^{n}R_i^2 + \left(n-4\right)\bar{R}_A^4 \end{aligned} \tag{37}$$

Excess kurtosis is fourth standardized moment minus three.

$$\gamma_{K,p} = \frac{\mu_4}{\sigma_p^4} - 3. \textbf{ (Population excess kurtosis)} \tag{38}$$

With limited sample data, the unbiased sample excess kurtosis can be expressed as

$$\gamma_{K,s} = \frac{n-1}{(n-2)(n-3)}\left\{\frac{\frac{1}{n}\sum_{i=1}^{n}\left(R_i - \bar{R}_A\right)^4}{\left[\frac{1}{n}\sum_{i=1}^{n}\left(R_i - \bar{R}_A\right)^2\right]^2} - 3\right\}$$
$$= \frac{n-1}{(n-2)(n-3)}\left[(n+1)\frac{\frac{1}{n}\sum_{i=1}^{n}R_i^4 - \frac{4\bar{R}_A}{n}\sum_{i=1}^{n}R_i^3 + \frac{6\bar{R}_A^2}{n}\sum_{i=1}^{n}R_i^2 + (n-4)\bar{R}_A^4}{\left(\frac{1}{n}\sum_{i=1}^{n}R_i^2 - \bar{R}_A^2\right)^2} + 6\right]. \quad \textbf{(Sample excess kurtosis)} \quad (39)$$

If excess kurtosis is zero, then the distribution is said to be mesokurtic. The normal distribution and binomial distributions are mesokurtic. If excess kurtosis is positive, then the distribution is said to be leptokurtic. Leptokurtic distributions have sharper peaks and fatter tails. The lognormal distribution is leptokurtic as well as the Laplace distribution and the logistic distribution. If excess kurtosis is negative, then the distribution is said to be platykurtic. Platykurtic distributions have flatter peaks and thinner tails. The uniform distribution and Bernoulli distribution (p=1/2) are platykurtic.

The lowest value of excess kurtosis is –2 and the highest value of excess kurtosis is positive infinity. A coin flip has a –2 excess kurtosis, it does not have a central peak at all. A student-t distribution with four degrees of freedom has an infinite excess kurtosis.

Example ?. Title

Annualizing individual instrument statistics

Note that skewness and excess kurtosis are unitless measures and thus do not require adjustments to annualize. Traditionally, arithmetic averages are annualized with arithmetic compounding or

$$\bar{R}_{a,A} = m\bar{R}_A = \frac{R_A}{\Delta t}, \quad (40)$$

and the standard deviation is a function of square root of frequency or

$$\sigma_{a,s} = \sigma_s\sqrt{m} = \sigma_s\sqrt{\frac{1}{\Delta t}}.$$

Clearly, any statistics integrating both an average and a standard deviation, great care must be used as the average grows linearly with horizon whereas the standard deviation grows linearly with the square root of the horizon. See, for example, the Sharpe ratio below.

Example ?. Title

Portfolio statistics

We now introduce some important portfolio statistics.

Position value

There are many different perspectives on financial value. We focus here on issues related to instrument value as contrasted with portfolio value. Finally, we make a few remarks related to the all-important position value, assets under management.

Instrument value
For most of the position within typical portfolios, there is an active secondary market and closing prices are directly observable. Taking a page from the accounting industry, instrument values can be categorized as one of three levels. The International Accounting Standards Board (IASB) and the Financial Accounting Standards Board (FASB) has issued promulgations on estimating fair value, including FASB's Accounting Standards Committee (ASC) 820, Fair Value Measurement (similarly the International Financial Reporting Standards (IFRS) 13, Fair Value Measurement).

Level 1 instrument values are based on quoted prices in active markets. For most instruments, this is the approach used and thus assumed. If for some reason, Level 1 fair value estimates are not available, then the analyst needs to document whether the fair value falls within Level 2 or 3 fair value estimation.

Level 2 instrument values will typically involve quoted prices of similar instruments in active markets, if available. If not available, then quoted identical instrument values where markets are inactive are used. Finally, instruments values are produced from inputs other than quoted prices, but are observable (e.g., interest rates, implied volatilities, and credit spreads).

Level 3 instrument values will involve inputs that are unobservable.

Clearly, confidence levels in the fair value estimate increase as we move from Level 1 to Level 2 to Level 3. Investment analysts should clearly denote instrument values that fall in Level 2 or 3 to alert users of this data that the fair value estimate contains greater potential for error.

Example ?. Title

Portfolio value
Net asset value (NAV) is the current value of assets less the current value of liabilities. For ETFs, it is simply the number of shares owned times the price summed over all investment instruments less any accrued liabilities. The value of your personal investment portfolio is typically reported as NAV. With portfolios that actively trade, such as ETFs, the NAV can differ from the ETF market price. Due to market forces, these deviations are often relatively small and can be exploited for traders with very low transaction costs.

In other markets, such as closed-end funds, the difference between the market price and NAV can differ substantially for a host of reasons. With closed-end funds, it is not generally possible to execute efficient trades to capture the difference.

Example ?. Title

Assets under management (AUM)
Assets under management (AUM) is a key metric for investment management firms. As the name suggests, AUM is simply the current fair value of the investment instruments currently under the discretion of a particular firm or manager. Investors when retaining wealth managers often grant these managers the right to buy and sell various instruments. Thus, the manager is said to have discretion over the funds.

Many firms charge clients a fee based on AUM, such as 80 basis points. There are many ways to calculate this fee. For example, the fee could be charged and collected quarterly. Thus, the charge would be 20 basis points per quarter. Further, the fee could be based on AUM and the beginning, the end, or even the daily average AUM for the quarter.

Thus, to accurately compute both the gross and net performance of any fund, the technical details of fees in terms of AUM are necessary. Though at times difficult to locate, the exact terms of management fees are required for many calculations.

Example ?. Title

Turnover ratio

The turnover ratio is a measure of active trading. If no shares of a portfolio were traded over a period, then the turnover ratio is zero. The turnover ratio is the percentage of a portfolio that has been replaced within a given period, typically one year.

The portfolio turnover ratio (*PTO*) can be represented as

$$PTO_t = \frac{MS}{ANA}100, \tag{41}$$

where *MS* denotes the minimum of instruments bought or sold and *ANA* denotes the average net assets in the portfolio. *MS* is the minimum of either the value of new instruments purchased, or the value of instruments sold over the given period. *ANA* is the average NAV in the fund.

For example, if a particular ETF held only 1,000 shares of a particular company, sold 300 shares, and purchased the same dollar amount of some other stock, then the turnover ratio is 30 percent. For actively traded portfolios, the turnover ratio can easily exceed 100 percent. Generally, the more active trading results in more transaction fees making it more difficult to generate excess performance.

Example ?. Title

Portfolio univariate statistics

Portfolio univariate statistics can be computed in the same way as individual instruments. The portfolio returns are used rather than an aggregation of individual instruments. For completeness, we repeat the formulas used. It is important to document whether the returns are gross or net of fees.

Mean

Sample arithmetic mean can be expressed as

$$\bar{R}_A = \frac{1}{n}\sum_{i=1}^{n} R_i. \textbf{ (Sample average)} \tag{42}$$

Sample geometric average can be expressed as

$$\bar{R}_G = \left[\prod_{i=1}^{n}\left(1+R_i\right)\right]^{\frac{1}{n}} - 1. \textbf{ (Geometric average)} \tag{43}$$

Example ?. Title

Standard deviation

Sample standard deviation can be expressed as

$$\sigma_s = \left(\frac{1}{n-1}\sum_{i=1}^{n} R_i^2 - \frac{n}{n-1}\bar{R}_A^2\right)^{1/2}. \textbf{ (Sample standard deviation)} \tag{44}$$

Example ?. Title

Skewness
Sample skewness is expressed as

$$\gamma_{S,s} = \frac{\sqrt{n(n-1)}}{n-2} \frac{\frac{1}{n}\sum_{i=1}^{n} R_i^3 - \frac{3\bar{R}_A}{n}\sum_{i=1}^{n} R_i^2 - (n-3)\bar{R}_A^3}{\left(\frac{1}{n}\sum_{i=1}^{n} R_i^2 - \bar{R}_A^2\right)^{3/2}}. \text{ (Sample skewness)} \tag{45}$$

Example ?. Title

Kurtosis
Sample excess kurtosis can be expressed as

$$\gamma_{K,s} = \frac{n-1}{(n-2)(n-3)}\left[(n+1)\frac{\frac{1}{n}\sum_{i=1}^{n} R_i^4 - \frac{4\bar{R}_A}{n}\sum_{i=1}^{n} R_i^3 + \frac{6\bar{R}_A^2}{n}\sum_{i=1}^{n} R_i^2 + (n-4)\bar{R}_A^4}{\left(\frac{1}{n}\sum_{i=1}^{n} R_i^2 - \bar{R}_A^2\right)^2} + 6\right]. \tag{46}$$

(Sample excess kurtosis)

Example ?. Title

Excess return
Excess return calculations require some reference instrument, such as a particular interest rate. A portfolio performance can be measured as a rate of return in excess of this reference instrument. We first examine the interest rate choice.
Interest rate choice
There are numerous options when the analysis requires some form of benchmark interest rate. Often these rates are termed the risk-free interest rate; although, we all know in life there is nothing free of risk. Unfortunately, each candidate benchmark interest rate suffers from at least one defect. Two dimensions related to the selection of a benchmark interest rate are maturity and type.
Maturity
By its very nature, an interest rate quotation has some maturity. Potential maturities include overnight (daily), weekly, monthly, annually, and various years to maturity (e.g., 2, 5, 10, 30, and perpetual).
Type
One common choice for the benchmark interest rate is government based, such as U. S. Treasury instruments. U. S. Treasury instruments include Treasury bills with maturities up to one year when issued, Treasury notes with maturities from two to ten years when issued (e.g., 2, 3, 5, 7, and 10 years), and U. S. Treasury bonds with maturities from 20 to 30 years when issued (e.g., 20 and 30 year). Bills are quoted on a discount basis whereas notes and bonds are quoted on a semi-annual bond equivalent yield basis (explained below). Rather than rely on government-based rates, an alternative is the Secured Overnight Financing Rate (SOFR).[6] There are numerous other candidate rates that may emerge over time, including the overnight index swap rate (OIS), the American interbank offered rate (Ameribor), and the Bloomberg short term bank yield index (BSBY). It is expected that over time there will be many other candidate rates available.

[6] For more information, see https://www.newyorkfed.org/markets/reference-rates/sofr-averages-and-index.

Example ?. Title

Quotation conventions

Great care must be given to managing interest rate quotations and resulting inferences. Let *NAD* denote the number of accrued days between trade settlement (usually T + 1 or trading day plus one trading day) and final maturity.

U. S. Treasury bills are quoted on a discount basis, so the bill value is

$$V_B = Par\left[1 - d\frac{NAD}{360}\right], \tag{47}$$

where d denotes the annualized discount rate, that is, the value at expiration, *Par*, is discounted by a rate tied to the number of accrued days (*NAD*) divided by 360 day year.

Solving for the discount rate, d, we have

$$d = \left(1 - \frac{V_B}{Par}\right)\frac{360}{NAD}. \tag{48}$$

Clearly, this method assumes a 360 day year and the quoted price is a discount to *Par*. To correct for these problems, the bond equivalent yield, y, is based on

$$y = \left(\frac{V_B}{Par} - 1\right)\frac{365}{NAD}. \tag{49}$$

For example, if Par = \$100,000, NAD = 100, and d = 1.878, then we have

$$V_B = Par\left[1 - d\frac{NAD}{360}\right] = 100{,}000\left[1 - 0.01878\frac{100}{360}\right] = 99{,}478.33.$$

Thus, the bond equivalent yield is

$$y = \left(\frac{Par}{V_B} - 1\right)\frac{365}{NAD} = \left(\frac{100{,}000}{99{,}478.33} - 1\right)\frac{365}{100} = 0.01914 \text{ or } 1.914\%.$$

Selected money market instruments, such as Eurodollar deposits are quoted on an add-on interest rate basis (denote a). Interbank Eurodollar deposits are the instruments on which LIBOR is based. The initial deposit value, V_B, is

$$Par = V_B\left(1 + a\frac{NAD}{360}\right) \text{ or} \tag{50}$$

$$V_B = \frac{Par}{1 + a\frac{NAD}{360}} \text{ or} \tag{51}$$

Solving for the add-on rate, a, we have

$$a = \left(\frac{Par}{V_B} - 1\right)\frac{360}{NAD}. \tag{52}$$

Again, this method assumes a 360 day year and the par value based on add on to the deposit amount, V_B. To convert to a bond equivalent yield, y, we have

$$y = \left(\frac{Par}{V_B} - 1 \right) \frac{365}{NAD}. \quad (53)$$

For example, if *Par* = \$100,000, *NAD* = 100, and *a* = 1.878, then we have

$$V_B = \frac{100,000}{1 + 0.01878 \frac{100}{360}} = 99,481.04.$$

Recall with the same discount yield, we had \$99,478.33 or an unacceptably high \$2.71 discrepancy. Thus, the bond equivalent yield is

$$y = \left(\frac{Par}{V_B} - 1 \right) \frac{365}{NAD} = \left(\frac{100,000}{99,481.04} - 1 \right) \frac{365}{100} = 0.01904 \text{ or } 1.904\%.$$

Recall the bond equivalent yield with the same discount rate was 1.9014% or a 1 basis point error. Although this error may seem small, over time errors of this nature can impair the integrity of reported performance.

Example ?. Title

Risk-free rate

It is vital that the interest rate reporting method is clear and conversion to total return is accurate. We term this series of total returns as the risk-free returns as they proxy a relatively riskless interest rate. Once a total return series is constructed, denoted V_t, then the period risk-free rate of return can be computed simply as

$$R_{dc,f} = R_f = \frac{V_{t+\Delta t}}{V_t} - 1. \quad (54)$$

Unless explicitly stated, we suppress the discrete compounding and simply use R_f.

Example ?. Title

Excess return (ER)

Excess returns are computed relative to some preselected reference instrument, such as the S&P 500 or the Russell 1000. We term this preselected reference instrument as the bogey. We avoid the use of benchmark as it carries a different meaning in another context here. There are two excess return measures, arithmetic and geometric.

The arithmetic excess return (*AER*) is simply

$$AER = R_I - R_B, \quad (55)$$

Where R_I denotes the rate of return of some instrument and R_B denotes the rate of return of the bogey. These rates of return need to be computed in the same manner, such as the arithmetic average of 60 monthly discretely compounded rates of return.

The geometric excess return (*GER*) is simply

$$GER = \frac{1 + R_I}{1 + R_B} - 1. \quad (56)$$

The geometric excess return is used for alpha calculation.

Example ?. Title

Sharpe Ratio (ShR)

The Sharpe ratio (*ShR*) measures the risk-adjusted return of an instrument where risk is measured in terms of the standard deviation of the instrument's rates of return. If we assume the bogey is the risk-free rate, R_f, then the Sharpe ratio is simply the arithmetic excess return normalized by the instrument standard deviation or

$$ShR = \frac{\bar{R}_I - R_f}{\sigma_I}. \tag{57}$$

Typically, the Sharpe ratio is based on some specified history of average discretely compounded rates of return, such as the last 60 months. The estimated standard deviation also typically covers the same historical period using the same instrument rates of return.

Example ?. Title

Sortino Ratio (SoR)

The Sortino ratio (*SoR*) measures the risk-adjusted return of an instrument where risk is measured in terms of the downside deviation of the instrument's rates of return. Further, the Sortino ratio is like the Sharpe ratio except the bogey may be some value other than the risk free interest rate. The instrument downside deviation (σ_I^{DD}) can be formally defined as

$$\sigma_I^{DD} = \sqrt{\int_{-\infty}^{R_T} \left(R_T - R_I\right)^2 f\left(R_I\right) dR_I}. \tag{58}$$

With discrete data, we have

$$\sigma_I^{DD} = \left\{ \frac{1}{n-1} \sum_{i=1}^{n} \left[\min\left(0, R_i - R_T\right)\right]^2 \right\}^{1/2}. \tag{59}$$

If we assume the bogey is some prespecified target rate, R_T, then the Sortino ratio is simply the arithmetic excess return normalized by the instrument downside deviation or

$$SoR = \frac{\bar{R}_I - R_T}{\sigma_I^{DD}}. \tag{60}$$

Typically, the Sortino ratio is based on some specified history of average discretely compounded rates of return, such as the last 60 months. The estimated downside deviation also typically covers the same historical period using the same instrument rates of return. The downside deviation, however, is often computed by calibrating to some statistical distribution, such as the normal distribution, given the limited number of downside observations.

Example ?. Title

Capture ratio

The capture ratio (CR) is based on the upside capture ratio (UCR) and the downside capture ratio (DCR). Recall I denotes the instrument being appraised and B denotes the corresponding bogey, and the cumulative annualized return (CAR) in this context for these two investment vehicles can be expressed as

$$CAR_I = \left(\prod_{i=1}^{n} TR_{I,i} \right)^{1/[n(\Delta t)]} - 1, \tag{61}$$

$$CAR_B = \left(\prod_{i=1}^{n} TR_{B,i} \right)^{1/[n(\Delta t)]} - 1, \tag{62}$$

The upside capture annualized return ($UCAR$) incorporates only the total return only when the *bogey* total return is greater than one or

$$UCAR_I = \left(\prod_{i=1}^{n} \begin{matrix} TR_{I,i} & \text{if } TR_{B,i} > 1 \\ 1 & \text{if } TR_{B,i} \le 1 \end{matrix} \right)^{1/[n(\Delta t)]} - 1 \text{ and} \tag{63}$$

$$UCAR_B = \left(\prod_{i=1}^{n} \begin{matrix} TR_{B,i} & \text{if } TR_{B,i} > 1 \\ 1 & \text{if } TR_{B,i} \le 1 \end{matrix} \right)^{1/[n(\Delta t)]} - 1. \tag{64}$$

The downside capture annualized return ($DCAR$) incorporates only the total return only when the *bogey* total return is less than one or

$$DCAR_I = \left(\prod_{i=1}^{n} \begin{matrix} TR_{I,i} & \text{if } TR_{B,i} < 1 \\ 1 & \text{if } TR_{B,i} \ge 1 \end{matrix} \right)^{1/[n(\Delta t)]} - 1 \text{ and} \tag{65}$$

$$DCAR_B = \left(\prod_{i=1}^{n} \begin{matrix} TR_{B,i} & \text{if } TR_{B,i} < 1 \\ 1 & \text{if } TR_{B,i} \ge 1 \end{matrix} \right)^{1/[n(\Delta t)]} - 1. \tag{66}$$

Thus, the upside and downside capture ratios can be expressed simply as

$$UCR_I = \frac{UCAR_I}{UCAR_B} \text{ and} \tag{67}$$

$$DCR_I = \frac{DCAR_I}{DCAR_B}. \tag{68}$$

Finally, the capture ratio is

$$CR_I = \frac{UCR_I}{DCR_I}. \tag{69}$$

Example ?. Title

Maximum drawdown (MD)

The maximum drawdown (MD) is a measure of an instrument's largest loss of value from its peak value (PkV) and its trough value (ThV). Maximum drawdown is one measure of downside risk and may be useful when comparing one instrument to another, such as a bogey. Maximum drawdown does not account for how frequently losses occur nor does it address the potential for gains.

When computing maximum drawdown, it is important to have a clearly specified period and to make sure that the peak value occurs prior to the trough value. The maximum drawdown is

$$MD = \frac{ThV_{\tau} - PkV_{\tau';\tau'<\tau}}{PkV_{\tau';\tau'<\tau}}, \tag{70}$$

Where t denotes the calendar time (say measured in years from the start of the specified period) when the trough value is observed and t' denotes the calendar time when the peak value is observed. Note that we require $t' < t$, that is, the peak value occurs prior to the trough value.

Example ?. Title

Relative portfolio performance measures

We compare the performance of a financial instrument to some prespecified bogey or possibly some prespecified market portfolio.

Tracking error (TE) and tracking risk (TR)

Tracking error is simply the difference between periodic rates of return of an instrument and a designated bogey. Tracking error is a measure of investment managers' active decision with respect to their bogey. Tracking error over Δt is simply

$$TE_{\Delta t} = R_{I,\Delta t} - R_{B,\Delta t}. \tag{71}$$

Typically, the tracking error is measured over some specified history of average discretely compounded rates of return, such as the last 60 months. The estimated tracking error also typically covers the same historical period using the instrument rates of return as well as the bogey rates of return. The estimated benchmark tracking risk (BTR) is computed in terms of standard deviation as

$$\begin{aligned} TR = \sigma_{BTR} &= \sigma\left(R_{I,\Delta t} - R_{B,\Delta t}\right) \\ &= \left[\frac{1}{n-1}\sum_{i=1}^{n}\left(R_{I,i} - R_{B,i}\right)^2 - \frac{n}{n-1}\left(\bar{R}_I - \bar{R}_B\right)\right]^{1/2}. \end{aligned} \tag{72}$$

We adopt the industry standard of using "tracking error" to mean the benchmark tracking risk.

Example ?. Title

Information ratio

The information ratio (InR) measures the risk-adjusted return of an instrument where risk is measured in terms of the standard deviation of the difference between the instrument's rates of return and the bogey's rate of return. This standard deviation measure is known as active risk or benchmark (bogey) tracking risk defined above.

If we assume the bogey is some prespecified benchmark index, R_B, then the information ratio is simply the arithmetic excess return normalized by the instrument downside deviation or

$$InR = \frac{\bar{R}_I - \bar{R}_B}{TR}. \tag{73}$$

Jensen's alpha and Treynor's index require the calculation of beta.

Example ?. Title

Beta (β)

Beta can be computed in many ways depending on the context. There is a distinct difference between beta with respect to the bogey (β_{IB}) and beta with respect to the preselected market instrument (β_{IM}). Regardless of which beta is being estimated, the general procedure is the same. The present the beta with respect to the bogey as

$$\beta_{IB} = \frac{\text{cov}\left(R_I, R_B\right)}{\sigma_I^2}, \tag{74}$$

where the variance is computed following the procedure above and the covariance is computed as

$$\text{cov}\left(R_I, R_B\right) = \frac{1}{n-1}\sum_{i=1}^{n} R_{I,i} R_{B,i} - \frac{n}{n-1}\bar{R}_I \bar{R}_B. \tag{75}$$

The beta with respect to the market instrument is simply the above two equations where the subscript B is replaced by M. A close cousin of beta is correlation—an often used measure of comovement.

Example ?. Title

Correlation (ρ)

Like beta, correlation can be computed in many ways depending on the context. There is a distinct difference between correlation with respect to the bogey (ρ_{IB}) and correlation with respect to the preselected market instrument (ρ_{IM}). Regardless of which correlation is being estimated, the general procedure is the same. The correlation with respect to the bogey as

$$\rho_{IB} = \frac{\text{cov}\left(R_I, R_B\right)}{\sigma_I \sigma_B}, \tag{76}$$

where the covariances and standard deviations are computed as above.

The correlation with respect to the market instrument is simply the above equation, where the subscript B is replaced by M.

Example ?. Title

R-Squared (R^2) or Coefficient of Determination

Like beta, R^2, also known as the coefficient of determination, can be computed in many ways depending on the context. There is a distinct difference between the R^2 with respect to the bogey (R_{IB}^2), the R^2 with respect to the preselected market instrument (R_{IM}^2), or some other R^2 with respect to some specified model. Regardless of which R^2 is being estimated, the general procedure is the same.

To illustrate, we assume a model where the financial instrument's rate of return (R_I) is a function (f) of the benchmark return (R_B) or

$$R_{I,t} = f(R_{B,t}) \text{ (the model).} \tag{77}$$

Recall the sample mean of the instrument's rate of return is $\bar{R}_I = \frac{1}{n}\sum_{t=1}^{n} R_{I,t}$. Let RSS denote the residual sum of squares of the model expressed as

$$RSS = \sum_{t=1}^{n} \left[R_{I,t} - f\left(R_{B,t} \right) \right]^2 . \tag{78}$$

Further, let TSS denote the total sum of squares of the model expressed as

$$TSS = \sum_{t=1}^{n} \left(R_{I,t} - \bar{R}_I \right)^2 . \tag{79}$$

The R^2 can be simply expressed as

$$R^2 = 1 - \frac{RSS}{TSS} . \tag{80}$$

R-squared and tracking risk (TR)
Tracking error is simply the difference between periodic rates of return of an instrument and a designated bogey. Tracking error is a measure of investment mangers' active decision with respect to their bogey. Tracking error over Δt is simply

$$TE_{\Delta t} = R_{I,\Delta t} - R_{B,\Delta t} . \tag{81}$$

Typically, the tracking error is measured over some specified history of average discretely compounded rates of return, such as the last 60 months. The estimated tracking error also typically covers the same historical period using the instrument rates of return as well as the bogey rates of return. The estimated benchmark tracking risk (BTR) is computed in terms of standard deviation as

$$\begin{aligned} TR = \sigma_{BTR} &= \sigma\left(R_{I,\Delta t} - R_{B,\Delta t} \right) \\ &= \left[\frac{1}{n-1} \sum_{i=1}^{n} \left(R_{I,i} - R_{B,i} \right)^2 - \frac{n}{n-1} \left(\bar{R}_I - \bar{R}_B \right) \right]^{1/2} . \end{aligned} \tag{82}$$

Jensen's alpha (α_J)
Jensen's alpha (α_J) is a performance index measured in percentage points above the instrument's predicted return based on some market model, typically the Capital Asset Pricing Model (CAPM). Jensen's alpha is

$$\alpha_J = R_I - \left[r_f + \beta_{IM} \left(R_M - r_f \right) \right] . \tag{83}$$

In this context, R_I denotes the realized rate of return vector on the instrument, R_M denotes the realized rate of return on the preselected market portfolio, r_f denotes the realized rate of return on the risk-free instrument, and β_{IM} denotes the beta with respect to the preselected market instrument.

Jensen's alpha measures the excess return over the predicted CAPM return based on some selected periodicity, some selected historical period, and some weighting scheme (discussed below).

Example ?. Title

Treynor's index (TI)
The Treynor ratio (TI) is a performance ratio measured in percentage points above the risk-free rate normalized by the instrument's beta to the prespecified market portfolio. The Treynor ratio is similar in structure the Sharpe ratio and is

$$TI = \frac{R_I - r_f}{\beta_{IM}} . \tag{84}$$

The Treynor ratio measures again is based on some selected periodicity, some selected historical period, and some weighting scheme (discussed below).

Example ?. Title

Portfolio Decomposition

In this section, we review the analytics related to a portfolio's underlying instruments.

Percentage marginal contribution to risk

For a portfolio (where Π emphasizes our focus now is on portfolios, capital Greek pi), the value of the portfolio can be represented as

$$\Pi_t = \sum_{j=1}^{n} N_{j,t} P_{j,t}, \tag{85}$$

where $N_{j,t}$ is the number of units of security j owned at time t (e.g., shares of stock) and $P_{j,t}$ is the value of one unit of security j observed at time t (e.g., stock price per share). Let n denote the total number of instruments owned at point in time t. The percentage rate of return over period t (length of time unspecified, formally from time t to time $t + 1$) can be expressed as:

$$R_{\Pi,t} = \sum_{j=1}^{n} w_{j,t} R_{j,t}, \text{ (Portfolio return decomposition)} \tag{86}$$

where $w_{j,t}$ denotes the proportion of portfolio Π invested in security j at point in time t and $R_{j,t}$ denotes the rate of return on security j over period t. Formally,

$$w_{j,t} = \frac{N_{j,t} P_{j,t}}{\sum_{j=1}^{n} N_{j,t} P_{j,t}} \text{ and} \tag{87}$$

$$R_{j,t} = \frac{P_{j,t+1}}{P_{j,t}} - 1. \tag{88}$$

The total risk of the portfolio can be measured as the variance of the rate of return on the portfolio or

$$\sigma_{\Pi}^2 \equiv E\left\{\left[\tilde{R}_{\Pi} - E\left(\tilde{R}_{\Pi}\right)\right]^2\right\} = \operatorname{cov}\left(\tilde{R}_{\Pi}, \tilde{R}_{\Pi}\right) = \operatorname{cov}\left(\sum_{j=1}^{n} w_j \tilde{R}_j, \tilde{R}_{\Pi}\right). \tag{89}$$

Recall covariance [cov()] is the expectation of the product of the deviations from the mean of which variance is a special case. The last equality in the equation above is a direct substitution from the portfolio return decomposition equation. From the properties of covariance, we can express the equation above as:

$$\sigma_{\Pi}^2 = Cov\left(\sum_{j=1}^{n} w_j \tilde{R}_j, \tilde{R}_{\Pi}\right) = \sum_{j=1}^{n} w_j Cov\left(\tilde{R}_j, \tilde{R}_{\Pi}\right). \tag{90}$$

Thus the marginal contribution to risk of any given security j (denote $MCTR_j$) within a portfolio is:

$$MCTR_j = w_j Cov\left(\tilde{R}_j, \tilde{R}_{\Pi}\right). \tag{91}$$

Clearly the sum of $MCTR_j$ is equal to the portfolio variance. Also note that if the covariance is negative, the contribution to risk is negative (assuming a long position or $w_j > 0$). Dividing both sides of the variance equation above by the portfolio variance we can compute the percentage marginal contribution to risk as:

$$1 = \frac{\sum_{j=1}^{n} MCTR_j}{\sigma_\Pi^2} = \frac{\sum_{j=1}^{n} w_j Cov\left(\tilde{R}_j, \tilde{R}_\Pi\right)}{\sigma_\Pi^2} = \sum_{j=1}^{n} w_j \beta_{j,\Pi} = \sum_{j=1}^{n} \%MCTR_j. \quad (92)$$

Therefore,

$$\%MCTR_j = w_j \beta_{j,\Pi}, \quad (93)$$

where

$$\beta_{j,\Pi} = \frac{\text{cov}\left(R_j, R_\Pi\right)}{\sigma_\Pi^2}. \quad (94)$$

Recall beta can be estimated as the slope of an ordinary least squares regression. The total risk as measured by variance can be decomposed into the risk related to each security.

Therefore, the percentage marginal contribution to risk measures the percentage of the variance attributable to a specific position in the portfolio. The sum of the percentage marginal contribution to risk is one.

$$\%MCTR_j = w_j \beta_{j,\Pi}. \quad (95)$$

Example ?. Title

Percentage marginal contribution to average excess return (develop generic predicted return models)

For a portfolio, the expected excess return can be represented as

$$E\left(\tilde{R}_\Pi\right) - r = \sum_{j=1}^{n} w_j E\left(\tilde{R}_j\right) - r = \sum_{j=1}^{n} w_j \left[E\left(\tilde{R}_j\right) - r\right]. \text{ (Portfolio expected excess return)} \quad (96)$$

Thus the marginal contribution to expected excess return of any given security j (denote $MCEERj$) within a portfolio is:

$$MCEER_j = w_j \left[E\left(\tilde{R}_j\right) - r\right]. \quad (97)$$

Clearly the sum of $MCEERj$ is equal to the portfolio's expected excess return. Also note that if the expected return is less than the risk free rate, the contribution to expected excess return is negative (assuming a long position or $w_j > 0$). Dividing both sides of the portfolio expected excess return equation by the portfolio's expected excess return, we can compute the percentage marginal contribution to expected excess return as:

$$1 = \frac{\sum_{j=1}^{n} MCEER_j}{E\left(\tilde{R}_\Pi\right) - r} = \frac{\sum_{j=1}^{n} w_j \left[\left(\tilde{R}_j\right) - r\right]}{E\left(\tilde{R}_\Pi\right) - r} = \sum_{j=1}^{n} w_j \frac{E\left(\tilde{R}_j\right) - r}{E\left(\tilde{R}_\Pi\right) - r} = \sum_{j=1}^{n} \%MCEER_j. \quad (98)$$

Therefore, the percentage marginal contribution to expected excess return measures the percentage of the expected excess return over the risk free interest rate attributable to a specific position in the portfolio. The sum of the percentage marginal contribution to expected excess return is one.

$$\%MCEER_j = w_j \frac{E\left(\tilde{R}_j\right)-r}{E\left(\tilde{R}_\Pi\right)-r}. \qquad (99)$$

Thus, we see that the total expected excess return can be decomposed into the expected excess return related to each security and measured on a percentage basis.

Example ?. Title

Return attribution analysis

Performance attribution is typically focused on decomposing the excess return (defined above). There are many ways to perform return attribution analysis. Here return attribution allocates a portion of the measured alpha into three categories, sector allocation decision (*SAD*), security selection decision (*SSD*), and an interaction term (*I*). After reviewing selected notation, we provide key results.

I – indicator function,
N – number of shares,
P – price per share,
Π – Managed portfolio,
B – Bogey portfolio,
k – sector,
t – period, and
SAW – sector allocation weight,

Key results

We review the main results of return attribution in this section. First, we identify one raw attribution measure. Excess sector allocation weights (*EW*) by subperiod and individual sector

$$EW_{k,t} = \left(SAW_{\Pi,k,t} - SAW_{B,k,t}\right); t,k. \qquad (100)$$

Selected key relationships:

$$\sum_{k=1}^{K}\left(SAW_{\Pi,k,t} - SAW_{B,k,t}\right) = 0 \text{ sum of weights equal to 1, total difference is 0,} \qquad (101)$$

$$SAW_{\Pi,k,t} = \sum_{j=1}^{n_M} w_{\Pi,j,t} I_{\text{Sector}_k}\left(j\right) \text{ weight of managed portfolio allocated to sector k,} \qquad (102)$$

$$SAW_{B,k,t} = \sum_{j=1}^{n_B} w_{B,j,t} I_{\text{Sector}_k}\left(j\right) \text{ weight of benchmark portfolio allocated to sector k,} \qquad (103)$$

$$w_{\Pi,j,t} = \frac{N_{\Pi,j,t} P_{\Pi,j,t}}{\sum_{j=1}^{n_M} N_{\Pi,j,t} P_{\Pi,j,t}} \text{ weight of managed portfolio allocated to instrument j, and} \qquad (104)$$

$$w_{B,j,t} = \frac{N_{B,j,t} P_{B,j,t}}{\sum_{j=1}^{n_B} N_{B,j,t} P_{B,j,t}} \text{ weight of benchmark portfolio allocated to individual instrument j.} \qquad (105)$$

We turn now to review our return attribution results. Excess sector returns (*ER*) by subperiod and individual sector is

$$ER_{k,t} = \left(R_{\Pi,k,t} - R_{B,k,t}\right); t,k. \qquad (106)$$

Sector allocation decision (*SAD*) by subperiod and individual sector is

$$SAD_{k,t} = \left(SAW_{\Pi,k,t} - SAW_{B,k,t}\right) R_{B,k,t}. \tag{107}$$

Security selection decision (*SSD*) by subperiod and individual sector is

$$SSD_{k,t} = SAW_{B,k,t}\left(R_{\Pi,k,t} - R_{B,k,t}\right). \tag{108}$$

Interaction (*I*) term by subperiod and individual sector is

$$I_{k,t} = \left(SAW_{\Pi,k,t} - SAW_{B,k,t}\right)\left(R_{\Pi,k,t} - R_{B,k,t}\right). \tag{109}$$

We now review portfolio properties based on discretely compounded returns.

Discrete return properties

$$\tilde{R}_{j,t} = \frac{P_{j,t+1}}{P_{j,t}} - 1 \text{ discretely compounded rate of return over time t for instrument j,} \tag{110}$$

$$R_{\Pi,t} = \sum_{j=1}^{n_\Pi} w_{\Pi,j,t} R_{j,t} \text{ rate of return over time t for the managed portfolio, and} \tag{111}$$

$$R_{B,t} = \sum_{j=1}^{n_B} w_{B,j,t} R_{j,t} \text{ rate of return over time t for the benchmark portfolio.} \tag{112}$$

Excess return is only additive across time (t) when forced with a time series adjustment term. The decomposition terms (sector allocation decision *SAD*, security selection decision *SSD*, and interaction *I*) are additive across sectors (k) at a point in time:

$$ER = SAD + SSD + I + I_{TS} = \sum_{t=1}^{T} ER_t = \sum_{t=1}^{T} SAD_t + \sum_{t=1}^{T} SSD_t + \sum_{t=1}^{T} I_t + I_{TS}, \tag{113}$$

where

$$SAD_t = \sum_{k=1}^{K} SAD_{k,t}, \tag{114}$$

$$SSD_t = \sum_{k=1}^{K} SSD_{k,t}, \text{ and} \tag{115}$$

$$I_t = \sum_{k=1}^{K} I_{k,t}. \tag{116}$$

We define the time series adjustment term as simply a plug figure to account for the non-additivity of the decomposition process or

$$I_{TS} = ER - SAD - SSD - I = \sum_{t=1}^{T} ER_t - \sum_{t=1}^{T} SAD_t - \sum_{t=1}^{T} SSD_t - \sum_{t=1}^{T} I_t. \tag{117}$$

Hence for each measurement period (e.g., a quarter), reported statistics include excess return (by subperiods and by sectors) and allocation decisions (also by subperiods and by sectors).

$$ER_t = SAD_t + SSD_t + I_t = \sum_{k=1}^{K}\left(SAW_{\Pi,k,t} - SAW_{B,k,t}\right)R_{B,k,t}$$
$$+\sum_{k=1}^{K} SAW_{B,k,t}\left(R_{\Pi,k,t} - R_{B,k,t}\right) + \sum_{k=1}^{K}\left(SAW_{\Pi,k,t} - SAW_{B,k,t}\right)\left(R_{\Pi,k,t} - R_{B,k,t}\right). \qquad (118)$$
$$= \sum_{k=1}^{K} SAW_{\Pi,k,t}R_{\Pi,k,t} - \sum_{k=1}^{K} SAW_{B,k,t}R_{B,k,t}$$

Appendix C provided more details on the justification for various return attribution results presented above.

Example ?. Title

Risk attribution analysis

Risk attribution decomposes the total variance of excess return into a variety of statistics including percentage marginal contribution to risk by stock, sector, sector allocation decision, security selection decision, and interaction.

As before,

$$R_\Pi = \sum_{j=1}^{n_\Pi} w_{\Pi,j}R_j = \sum_{k=1}^{K} SAR_{\Pi,k} = \sum_{k=1}^{K} SAW_{\Pi,k}R_{\Pi,k} \text{ return on managed portfolio,} \qquad (119)$$

$$SAR_{\Pi,k} = SAW_{\Pi,k}R_{\Pi,k} \text{ managed, weight-adjusted, sector allocation return,} \qquad (120)$$

$$R_B = \sum_{j=1}^{n_B} w_{B,j}R_j = \sum_{k=1}^{K} SAR_{B,k} = \sum_{k=1}^{K} SAW_{B,k}R_{B,k} \text{ return on benchmark portfolio, and} \qquad (121)$$

$$SAR_{B,k} = SAW_{B,k}R_{B,k} \text{ benchmark, weight-adjusted, sector allocation return.} \qquad (122)$$

Extensive use of uniquely constructed beta coefficients is applied here, including

$$\beta_{j,\Pi} = \frac{\operatorname{cov}\left(R_j, R_\Pi\right)}{\sigma_\Pi^2} \text{ beta of } j^{\text{th}} \text{ instrument with the portfolio,} \qquad (123)$$

$$\beta_{\Pi,ER,k} = \frac{\operatorname{cov}\left(R_{\Pi,k}, ER\right)}{\sigma_{ER}^2} \text{ beta of the } k^{\text{th}} \text{ sector of the managed portfolio with } ER, \qquad (124)$$

$$\beta_{B,ER,k} = \frac{\operatorname{cov}\left(R_{B,k}, ER\right)}{\sigma_{ER}^2} \text{ beta of the } k^{\text{th}} \text{ sector of the benchmark portfolio with } ER, \qquad (125)$$

$$\beta_{SAD,ER} = \frac{\operatorname{cov}\left(SAD, ER\right)}{\sigma_{ER}^2} \text{ beta of the } SAD \text{ component with excess return,} \qquad (126)$$

$$\beta_{SSD,ER} = \frac{\operatorname{cov}\left(SSD, ER\right)}{\sigma_{ER}^2} \text{ beta of the } SSD \text{ component with excess return, and} \qquad (127)$$

$$\beta_{I,ER} = \frac{\operatorname{cov}\left(I, ER\right)}{\sigma_{ER}^2} \text{ beta of the interaction component with excess return.} \qquad (128)$$

Based on analysis presented in Appendix D, we have the following key results.

Key statistics

$$\beta_{SAD,ER,k} = \left(SAW_{\Pi,k} - SAW_{B,k}\right)\beta_{B,k,ER}, \tag{129}$$

$$\beta_{SSD,ER,k} = SAW_{B,k}\left(\beta_{\Pi,k,ER} - \beta_{B,k,ER}\right), \text{ and} \tag{130}$$

$$\beta_{I,ER,k} = \left(SAW_{\Pi,k} - SAW_{B,k}\right)\left(\beta_{\Pi,k,ER} - \beta_{B,k,ER}\right). \tag{131}$$

Variance properties

$$\begin{aligned}\sigma^2\left(R_\Pi\right) &= \sum_{j=1}^{n_\Pi} w_{\Pi,j}\operatorname{cov}\left(R_j, R_\Pi\right) \\ &= \sum_{k=1}^{K}\operatorname{cov}\left(SAR_{\Pi,k}, R_\Pi\right) \\ &= \sum_{k=1}^{K} SAW_{\Pi,k}\operatorname{cov}\left(R_{\Pi,k}, R_\Pi\right)\end{aligned}, \tag{132}$$

$$\begin{aligned}\sigma^2\left(R_B\right) &= \sum_{j=1}^{n_B} w_{B,j}\operatorname{cov}\left(R_j, R_B\right) \\ &= \sum_{k=1}^{K}\operatorname{cov}\left(SAR_{B,k}, R_B\right) \\ &= \sum_{k=1}^{K} SAW_{B,k}\operatorname{cov}\left(R_{B,k}, R_B\right)\end{aligned}, \text{ and} \tag{16.2.133}$$

$$\begin{aligned}\sigma^2\left(ER\right) &= \operatorname{cov}\left(R_\Pi - R_B, ER\right) \\ &= \sum_{j=1}^{n_\Pi} w_{\Pi,j}\operatorname{cov}\left(R_j, ER\right) - \sum_{j=1}^{n_B} w_{B,j}\operatorname{cov}\left(R_j, ER\right) \\ &= \operatorname{cov}\left(SAD + SSD + I, ER\right) \\ &= \operatorname{cov}\left(SAD, ER\right) + \operatorname{cov}\left(SSD, ER\right) + \operatorname{cov}\left(I, ER\right)\end{aligned}. \tag{134}$$

Betas are defined as

$$\beta_{j,\Pi} = \frac{\operatorname{cov}\left(R_j, R_\Pi\right)}{\sigma_\Pi^2} \text{ beta of } j^{\text{th}} \text{ instrument with the portfolio,} \tag{135}$$

$$\beta_{\Pi,ER,k} = \frac{\operatorname{cov}\left(R_{\Pi,k}, ER\right)}{\sigma_{ER}^2} \text{ beta of the } k^{\text{th}} \text{ sector of the managed portfolio with } ER\text{, and} \tag{136}$$

$$\beta_{B,ER,k} = \frac{\operatorname{cov}\left(R_{B,k}, ER\right)}{\sigma_{ER}^2} \text{ beta of the } k^{\text{th}} \text{ sector of the benchmark portfolio with } ER. \tag{137}$$

Note

$$1 = \frac{\sum_{j=1}^{n} MCTR_j}{\sigma_\Pi^2} = \frac{\sum_{j=1}^{n} w_j \operatorname{cov}\left(R_j, R_\Pi\right)}{\sigma_\Pi^2} = \sum_{j=1}^{n} w_j \beta_{j,\Pi} = \sum_{j=1}^{n} \% MCTR_j, \tag{138}$$

$$\% MCTR_j = w_j \beta_{j,\Pi} \text{ percentage marginal contribution to risk for instrument j,} \tag{139}$$

$$1 = \frac{\operatorname{cov}\left(R_\Pi - R_B, ER\right)}{\sigma_{ER}^2} = \frac{\operatorname{cov}\left(R_\Pi, ER\right)}{\sigma_{ER}^2} - \frac{\operatorname{cov}\left(R_B, ER\right)}{\sigma_{ER}^2} = \beta_{\Pi,ER} - \beta_{B,ER}$$

$$1 = \frac{\mathrm{cov}\left(SAD + SSD + I, ER\right)}{\sigma_{ER}^2} = \frac{\mathrm{cov}\left(SAD, ER\right)}{\sigma_{ER}^2} + \frac{\mathrm{cov}\left(SSD, ER\right)}{\sigma_{ER}^2} + \frac{\mathrm{cov}\left(I, ER\right)}{\sigma_{ER}^2}, \text{ and} \quad (140)$$

$$= \beta_{SAD,ER} + \beta_{SSD,ER} + \beta_{I,ER}$$

$$\beta_{SAD,ER} + \beta_{SSD,ER} + \beta_{I,ER} = \beta_{\Pi,ER} - \beta_{B,ER} = 1. \quad (141)$$

Appendix D provides further detail of this material

Example ?. Title

Statistical weighting

The statistical calculations can be made with a variety of weighting schemes. The standard approach is to equally weight each observation. There are many other approaches. We review here an exponential weighting and illustrate it over time; hence, we term it the exponentially weighted moving average.

Exponentially weighted moving average (EWMA)

Again, when estimating sample statistics, one may wish to weight observations differently. For example, when dealing with financial data, there may be an interest in weighing the more recent observations more heavily. The key is having the flexibility to apply different weighting schemes as needed.

One common approach is known as EWMA. As this application is usually applied to time series data, hence the term “moving.” With daily data, when one new day enters the data set, then we drop the oldest day and recalculate. Thus, we use the term EWMA. Applying this approach to the last N observations, the weighting scheme can be represented as

$$w_i = \frac{1-\lambda}{1-\lambda^N}\lambda^{i-1} \text{ (i = 1, …, N)}, \quad (142)$$

where N denotes the number of observations in the rolling window (e.g., past 60 months, $N = 60$), and λ ($0 < \lambda < 1$) influences the speed that the weight decline across the data. As λ declines from 1, the weights decline more rapidly.

It can be shown that the sum of the weights equals one. The following tables illustrate three different selections of λ for up to $N = 10$. More weight is applied to lower values of i (typically more recent observations) and with exponentially declining weight. Table 1 uses the common selection of $\lambda = 0.94$. Notice the rapid decline of the weights.

Table 1. Illustrations the Proportions Within an Exponentially Weighted Moving Average

Exponentially Weighted Moving Average											
Lambda	0.940000										
Weights\N	1	2	3	4	5	6	7	8	9	10	Change (10)
1	1.0000	0.5155	0.3542	0.2737	0.2255	0.1935	0.1707	0.1537	0.1405	0.1300	0.0078
2		0.4845	0.3329	0.2572	0.2120	0.1819	0.1604	0.1445	0.1321	0.1222	0.0073
3			0.3129	0.2418	0.1992	0.1709	0.1508	0.1358	0.1242	0.1149	0.0069
4				0.2273	0.1873	0.1607	0.1418	0.1276	0.1167	0.1080	0.0065
5					0.1760	0.1510	0.1333	0.1200	0.1097	0.1015	0.0061
6						0.1420	0.1253	0.1128	0.1031	0.0954	0.0057
7							0.1178	0.1060	0.0969	0.0897	0.0054
8								0.0997	0.0911	0.0843	0.0051
9									0.0857	0.0793	0.0048
10										0.0745	
Sum(W)	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	

When λ is set very close to 1 (say 0.999999), then we essentially have the traditional equally weighting scheme as seen in Table 2.

Table 2. Illustrations the Proportions Within an Exponentially Weighted Moving Average (λ = 0.999999)

Exponentially Weighted Moving Average											
Lambda	0.999999										
Weights\N	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**Change (10)**
1	1.0000	0.5000	0.3333	0.2500	0.2000	0.1667	0.1429	0.1250	0.1111	0.1000	0.0000
2		0.5000	0.3333	0.2500	0.2000	0.1667	0.1429	0.1250	0.1111	0.1000	0.0000
3			0.3333	0.2500	0.2000	0.1667	0.1429	0.1250	0.1111	0.1000	0.0000
4				0.2500	0.2000	0.1667	0.1429	0.1250	0.1111	0.1000	0.0000
5					0.2000	0.1667	0.1429	0.1250	0.1111	0.1000	0.0000
6						0.1667	0.1429	0.1250	0.1111	0.1000	0.0000
7							0.1429	0.1250	0.1111	0.1000	0.0000
8								0.1250	0.1111	0.1000	0.0000
9									0.1111	0.1000	0.0000
10										0.1000	
Sum(W)	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	

With lower λ values, then more weight is applied to lower values of i. That is, more weight is applied to recent observations assuming the data is sorted such that the most recent observation is number 1. For example, Table 3 illustrates $\lambda = 0.7$.

Table 3. Illustrations the Proportions Within an Exponentially Weighted Moving Average (λ = 0.7)

Exponentially Weighted Moving Average											
Lambda	0.700000										
Weights\N	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**Change (10)**
1	1.0000	0.5882	0.4566	0.3948	0.3606	0.3400	0.3269	0.3184	0.3126	0.3087	0.0926
2		0.4118	0.3196	0.2764	0.2524	0.2380	0.2288	0.2228	0.2188	0.2161	0.0648
3			0.2237	0.1934	0.1767	0.1666	0.1602	0.1560	0.1532	0.1513	0.0454
4				0.1354	0.1237	0.1166	0.1121	0.1092	0.1072	0.1059	0.0318
5					0.0866	0.0816	0.0785	0.0764	0.0751	0.0741	0.0222
6						0.0571	0.0549	0.0535	0.0525	0.0519	0.0156
7							0.0385	0.0375	0.0368	0.0363	0.0109
8								0.0262	0.0257	0.0254	0.0076
9									0.0180	0.0178	0.0053
10										0.0125	
Sum(W)	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	

Figure 4 illustrates the exponentially weighting scheme with $\lambda = 0.94$ applied to sixty observations.

Figure 4. Exponential Weighting for 60 Observations (⌧ = 0.94)

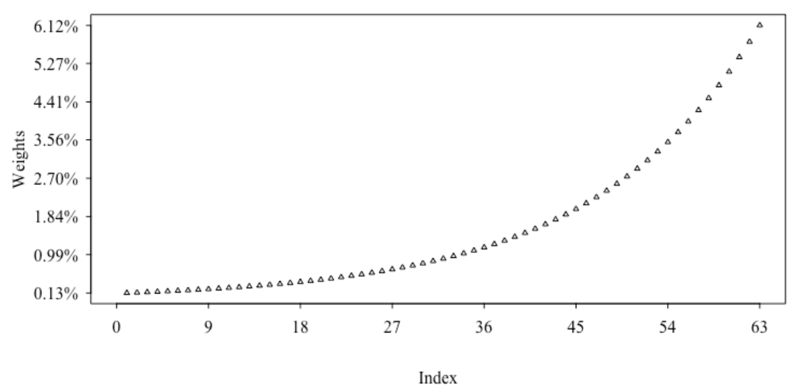


Figure 5 illustrates the means for these three funds based on EWMA ($\lambda = 0.94$) and 63 trading days (one quarter of a trading day year of 252) over this period. Rolling means appear very unstable with only of few discernable patterns. The benchmark (BMK or SPY) is more volatile than the high risk fund (HRF or HYG) and the low risk fund (LRF or TNT). There is mild evidence that the rolling mean of the LRF is negatively correlated with BMK. We will see this more clearly later.

Figure 5. EWMA Rolling Means for BMK, HRF, and LRF

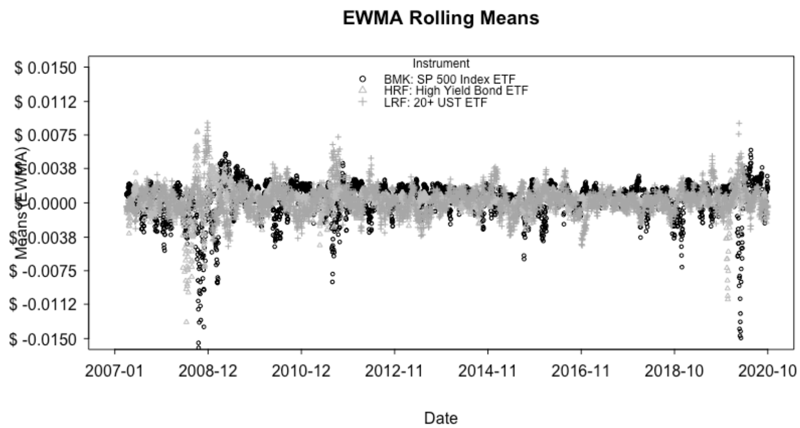


Figure 6 illustrates the standard deviation for these three funds based on EWMA ($\lambda = 0.94$) and 63 trading days (one quarter of a trading day year of 252) over this period. Notice that BMK is the riskiest whereas somewhat surprisingly HRF is the least risky. Further, LRF is quite risky. Recall HRF is highly diversified of disparate credit risks whereas LRF is essentially a single position in a long-term U.S. treasury.

Figure 6. EWMA Rolling Standard Deviations for BMK, HRF, and LRF

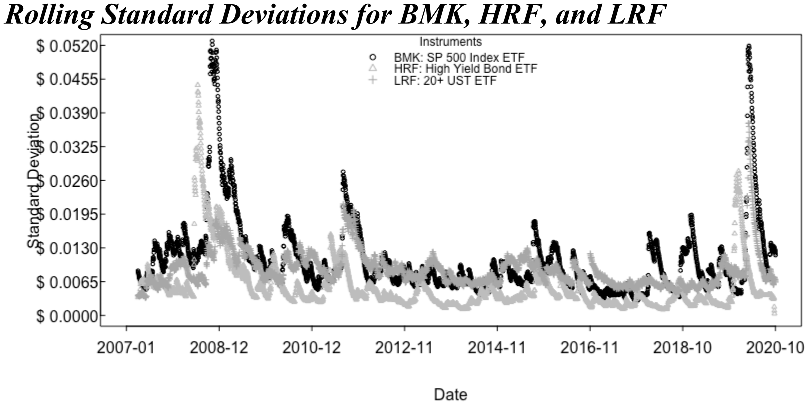
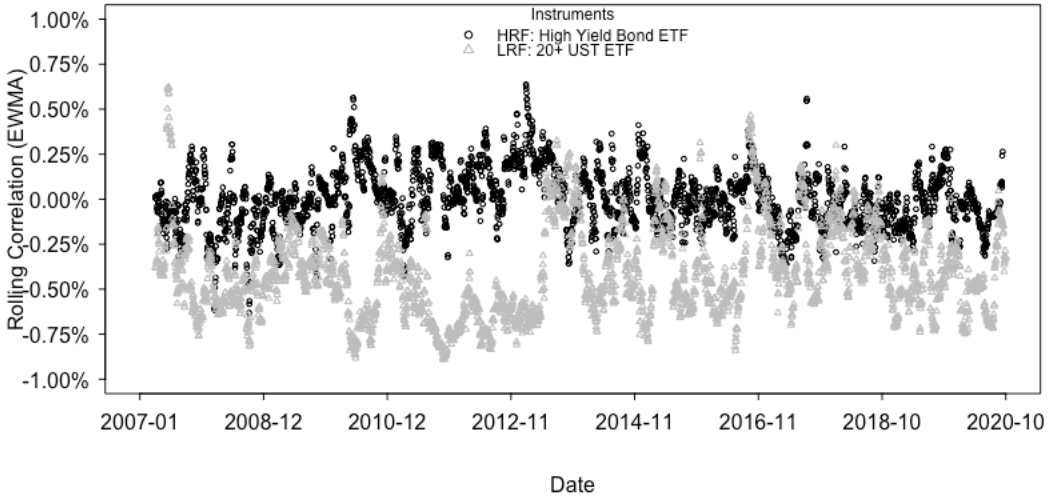


Figure 7 illustrates the correlation with SPY for these two bond funds based on EWMA ($\lambda = 0.94$) and 63 trading days (one quarter of a trading day year of 252) over this period. The key insight is that the LRF is typically negatively correlated with the equity BMK whereas the HRF is typically positively correlated with the BMK. Thus, LRF provides significant diversification benefits.

Figure 7. Rolling Correlations with BMK for HRF and LRF



Symbols used

A	Adjusted price (adjusted for cash dividends, splits, and other corporate actions)
α_J	Jensen's alpha
Ameribor	American interbank offered rate, a U.S. centric alternative to Libor
ANA	Average net assets in the portfolio
ASC	Accounting Standards Committee
AUM	Assets under management
$\beta_{i,\Pi}$	Beta (lower case), beta of instrument with managed portfolio
β_{IB}	Beta (lower case), beta of instrument with bogey
β_{IM}	Beta (lower case), beta of instrument with market portfolio
B	Bogey (an instrument used as a benchmark to appraise relative performance)
BMV	Beginning market value
BSBY	Bloomberg short term bank yield index, an alternative to Libor
BTR	Benchmark tracking risk
CAPM	Capital Asset Pricing Model, used to estimate an instrument's expected return
CAR	Cumulative annual return
$\text{cov}(x,y)$	Covariance between two variables, x and y
CR	Capture ratio
CRSP	Center for Research in Security Prices, source for historical financial instrument data
CSI	Commodity Systems, Inc., a low priced source for historical financial instrument data
d	Discount rate used with U.S. Treasury bills
D	Dollar dividend amount
$\hat{D}$	Implied dividend based on adjusted prices
DCAR	Downside capture annualized return
DCR	Downside capture ratio
Δt	Delta t, change in calendar time, typically a specified period such as a day, week, month, or year
EMV	Ending market value
$\text{E}(R)$	Expected rate of return (subjective or model dependent)
ER	Excess return
ETF	Exchange-traded fund
EW	Excess weight, used in performance attribution analysis
FASB	Financial Accounting Standards Board, promulgates U.S.-based accounting standards
$\gamma_{K,p}$	Gamma (lower case), population kurtosis
$\gamma_{K,s}$	Gamma (lower case), sample kurtosis
$\gamma_{S,p}$	Gamma (lower case), population skewness
$\gamma_{S,s}$	Gamma (lower case), sample skewness
GER	Geometric excess return
GIPS	Global Investment Presentation Standards, promulgated by the CFA Institute, provides uniformity on reporting instrument performance
I_t	Interaction term observed at time t
$I_{k,t}$	Interaction term of sector k observed at time t
I_{TS}	Interaction term capturing time series error in performance attribution analysis
IASB	International Accounting Standards Board
IFRS	International Financial Reporting Standards
Inc	Any positive cash flow (income) provided by an instrument
InR	Information ratio

IRoR	Interim rate of return
k	Counter for sectors, used in performance attribution analysis
λ	Lambda (lower case), used in exponentially weighting schemes, ($0 < \lambda < 1$) influences the speed that the weight decline across the data
Libor	London interbank offered rate, a measure of international bank's marginal cost of funds
m	Number of instruments
max(…)	Maximum of series of numbers
MD	Maximum drawdown
min(…)	Minimum of series of numbers
MCEER	Marginal contribution to expected excess return
MCTR	Marginal contribution to risk
MS	Minimum of instruments bought or sold
$\mu_1, \mu_2, \mu_3, \mu_4$	Mu (lower case), central moments, μ_1 is first moment of the distribution, remaining are central moments around mean
N	Number of past observations used when applying exponential weighting
$N_{j,t}$	Number of units (shares) of instrument j held at time t
n	Number of periods
NAD	Number of accrued days
NAV	Net asset value
OIS	Overnight index swap rate, an alternative to Libor
%MCEER	Percentage marginal contribution to expected excess return
%MCTR	Percentage marginal contribution to risk
P	Observed market price or estimated fair value of an instrument
$P_{j,t}$	Price or fair market value of instrument j held at time t
Par	Par value of a bond
PkV	Peak value of some instrument over some period
Π	Pi (upper case), symbol for product of a series of numbers
$P\bar{E}$	Harmonic average of price-earnings ratio
Profit	Gain in instrument value as well as any positive cash flow (income, *Inc*) provided
PTO	Portfolio turnover ratio
$\bar{R}_A$	Sample arithmetic average
$\bar{R}_{a,A}$	Annualized sample arithmetic average
$R_{a,cc}$	Annualized rate of return, continuously compounded
$R_{a,dc}$	Annualized rate of return, discretely compounded
R_B	For instrument B (bogey), periodic rate of return over period i, typically discretely compounded
R_{cc}	Periodic rate of return, continuously compounded
R_{dc}	Periodic rate of return, discretely compounded
$R_{dc,f}$, R_f	Risk-free interest rate, typically annualized, discretely compounded
$\bar{R}_G$	Sample geometric average
$\bar{R}_H$	Sample harmonic average
R_i	Periodic rate of return over period i, typically discretely compounded
$R_{j,t}$	Rate of return of instrument j held at time t
R_I	For instrument I, periodic rate of return, typically discretely compounded
R_M	For market portfolio, periodic rate of return, typically discretely compounded
$R_{\Pi,t}$	Rate of return of portfolio Π held at time t
R_T	Target periodic rate of return, typically discretely compounded
$\rho_{I,B}$	Rho (lower case), correlation coefficient between instrument and bogey
$\rho_{I,B}$	Rho (lower case), correlation coefficient between instrument and market portfolio

Σ	Sigma (upper case), symbol for summation
$\sigma_{a,s}$	Sigma (lower case), annualized sample standard deviation
σ_{BTR}	Sigma (lower case), sample standard deviation of benchmark tracking error
σ_I^{DD}	Sigma (lower case), for instrument I, sample standard deviation of the instrument downside deviation from the target return
σ_Π^2	Sigma (lower case), sample variance of the portfolio
σ_Π	Sigma (lower case), sample standard deviation of the portfolio
σ_p^2	Sigma (lower case), population variance
σ_p	Sigma (lower case), population standard deviation
σ_s^2	Sigma (lower case), sample variance
σ_s	Sigma (lower case), sample standard deviation
SAD_t	Sector allocation decision term observed at time t
$SAD_{k,t}$	Sector allocation decision term of sector k observed at time t
SAR	Sector allocation return
SAW	Sector allocation weight
ShR	Sharpe ratio
$SOFR$	Secured overnight financing rate, an alternative to Libor
SoR	Sortino ratio
SSD_t	Stock selection decision term observed at time t
$SSD_{k,t}$	Stock selection decision term of sector k observed at time t
τ, τ'	Tau (lower case), points in time measured in years over specified period, where $\tau < \tau'$
t	Point in calendar time, typically measured as a fraction of year since some initial date, e.g., $t = 0.5$
TE	Tracking error
TI	Treynor's index
TR	Tracking risk
$TR_{\Delta t}$	Total return over period Δt
ThV	Trough (lowest) value of some instrument over some period
$UCAR$	Upside capture annualized return
UCR	Upside capture ratio
V_B	Fair market value of an instrument designated as the bogey, observed at time t
V_t	Fair market value of an instrument observed at time t
$V_{t+\Delta t}$	Fair market value of an instrument observed at time $t + \Delta t$
$w_{j,t}$	Weight or proportion of instrument j held in a portfolio at time t
y	Yield to maturity or bond equivalent yield to maturity

Appendix A. Presuppositions and assumptions[7]

Though often not articulated, there are several important presuppositions and assumptions made with respect to investment management. As culture changes, particularly the presuppositions may be stressed; hence, key foundational ideas may shift altering the appropriate strategies.

Presuppositions

We suggest that there are at least four presuppositions for financial markets to reasonably function. A presupposition is a requirement that is antecedent in logic or fact, that is, it is what is assumed beforehand.

First, there needs to be clear rule of law. Ambiguity in law leads to tyranny in enforcement. For example, if the speed limit is set to be "reasonable," then any law enforcement officer can arrest anyone for speeding. The law enforcement officer can arbitrarily determine that your speed was unreasonable, especially if you are a member of the wrong political party.

Second, to execute a transaction, there needs to be clean property rights. If property ownership is uncertain, then buying or selling that property will result in disputes.

Case (2003) notes, "(T)he degree to which the society is bound by law, is committed to processes that allow property rights to be secure under legal rules that will be applied predictably and not subject to the whims of particular individuals, matters." (p. 2)

Third, financial markets are more efficient if built on a foundation of trust. Trust implies you rely on someone with something of value. If you trust, then you make yourself vulnerable in confidence. You are assuming the trusted will not exploit and will be concerned. For example, medical surgery would be impossible if doctors were not at least somewhat trustworthy. Clearly, trust makes cooperative activities, such as financial markets, possible.

Finally, we assume that the uncertainties related to future activities can reasonably be mapped to risk, meaning a subjective probability distribution of some form. This final presupposition is key to understanding why investments will always be within the social sciences and not the natural sciences. The mapping from uncertainty to risk will always entail epistemic uncertainty that cannot be eliminated. For example, estimating the future expected return remains subjective, there is no objective measure for it

Assumptions

We turn now to specific assumptions typically made regarding the nature of financial instrument price behavior. The goal is to convert what is inherently uncertainty and hence technically unmeasurable into risk and hence measurable.

Foundational assumptions

We review selected foundational assumptions typically present, though often unspecified, with financial models. For more exhaustive details, see Harrison and Kreps (1979) and Harrison and Pliska (1981).

1. $\left[0,\hat{T}\right]$, for fixed $0 \le t \le \hat{T}$, finite time horizon. Thus, calendar time can be expressed as a finite segment of the real number line.
2. $\left(\Omega,\Im,\mathrm{P}\right)$, uncertainty is characterized by a complete probability space, where the state space Ω is the set of all possible realizations of the stochastic economy between time 0 and time $\hat{T}$ and has a typical element ω representing a sample path, $\Im$ is the sigma field of distinguishable events at time $\hat{T}$, and P is a probability measure defined on the elements of $\Im$. (See more detailed explanation below.)
3. $F=\left[\Im(t):t\in\left(0,\hat{T}\right)\right]$ the augmented, right continuous, complete filtration generated by the appropriate stochastic processes in the economy and assume that $\Im\left(\hat{T}\right)=\Im$. The augmented

[7]See Chapter 2 in Robert Brooks, *Building Quantitative Finance Applications with R*, forthcoming, for references and more details. Used with permission.

filtration, $\Im(t)$, is generated by Z. $\Im(0)$ contains only Ω and the null sets of P. In a finance context, the filtration is keeping track of what is known at a point in time and what is not known.

4. F is generated by a K-dimensional Brownian motion, $Z(t)=\left[Z_1(t),\cdots,Z_K(t)\right],t\in\left(0,\hat{T}\right)$ is defined on $(\Omega,\Im,P)$, where $\left[\Im(t)\right],t\in\left(0,\hat{T}\right)$ is the augmentation of the filtration $\left[\Im^Z(t)\right],t\in\left(0,\hat{T}\right)$ generated by $Z(t)$, and satisfies the usual conditions.
5. $E_P(\cdot)$ denotes the expectation with respect to the probability measure P.
6. All stated equalities or inequalities involving random variables hold P-almost surely.
7. P is common for all agents implying uniqueness of the nature of the stochastic processes.
8. Conventional perfect market conditions are typically assumed, such as no transaction costs, no taxes, unrestricted short selling, and no regulatory or institutional constraints.
9. Future financial instrument values can be represented by some distribution.

We provide more details on a few of these assumptions.

- $(\Omega,\Im,P)$ characterizes uncertainty using a complete probability space, where the state space Ω denotes the set of all possible realizations between time 0 and time $\hat{T}$, ω represents one sample path, $\Im$ denotes the sigma field of events known at time $\hat{T}$, and P is a probability measure defined on the sigma field, $\Im$. $(\Omega,\Im,P)$ is a mathematical representation of our perceptions of unpredictable movements in underlying instrument prices
- Uncertainty means unpredictable change (both likelihood and outcome are unknown)
- Complete probability space – uncertainty is reduced to risk (both likelihood and outcome are known)
- Ω – state space, all possible sample paths representing a model of uncertainty
- 0 – time is measurable, our analysis is limited to a finite time length
- $\hat{T}$ – terminal point in time
- ω – a unique, particular events (e.g., sample path), known only at time $\hat{T}$
- $\Im$ – sigma field, a collection of sets illustrated below
- P – a probability measure defined on $\Im$

Consider a three period binomial illustration where each period is 1 year and the likelihood of up is 3/5ths:

- Ω – {ϕ, {d}, {u}, {d,d}, {d,u}, {u,d}, {u,u}, {d,d,d}, {d,d,u}, {d,u,d}, {d,u,u}, {u,d,d}, {u,d,u}, {u,u,d}, {u,u,u}}
- 0 – initial period in binomial illustration, $\hat{T}-3$
- ω – {u,d,u}, $\Im$ – keeps track of information (complete past sample path)
- t=0 {ϕ, {d}, {u}, {d,d}, {d,u}, {u,d}, {u,u}, {d,d,d}, {d,d,u}, {d,u,d}, {d,u,u}, {u,d,d},
{u,d,u}, {u,u,d}, {u,u,u}} (100%)
- t=1: {{d}, {d,d}, {d,u}, {d,d,d}, {d,d,u}, {d,u,d}, {d,u,u}} (40%)
{{u}, {u,d}, {u,u}, {u,d,d}, {u,d,u}, {u,u,d}, {u,u,u}} (60%)
- t=2: {{d,d}, {d,d,d}, {d,d,u}} (16%)
{{d,u}, {d,u,d}, {d,u,u}} (24%)
{{u,d}, {u,d,d}, {u,d,u}} (24%)
{{u,u}, {u,u,d}, {u,u,u}} (36%)
- t=3: {{d,d,d}} (6.4%)
{{d,d,u}} (9.6%)
{{d,u,d}} (9.6%)
{{d,u,u}} (14.4%)
{{u,d,d}} (9.6%)
{{u,d,u}} (14.4%)

{{u,u,d}} (14.4%)
{{u,u,u}} (21.6%)

Note the state space is the set of all possible realizations between time 0 and time $\hat{T}$. In this case, it is Ω or {ϕ, {d}, {u}, {d,d}, {d,u}, {u,d}, {u,u}, {d,d,d}, {d,d,u}, {d,u,d}, {d,u,u}, {u,d,d}, {u,d,u}, {u,u,d}, {u,u,u}}. The sigma field of distinguishable events keeps track of what is known at any point in time. Notice above that as each point in time passes, the set of possible distinguishable events is reduced.

Appendix B. Return calculations

We provide extensive details on various return calculations here.

Calculating rates of return[8]

Given the numerous ways rates of return could be reported, we provide background on the standard approach. As objectives vary including an array of holding periods, we provide details on the interim rate of return calculations.

Interim rate of return

The total rate of return over some period usually requires computing interim rates of return and linking them together in an appropriate manner. The total rate of return is simply the ending market value divided by the beginning market value minus one. Although relatively intuitive to describe, it is often incorrectly computed in practice. The interim rate of return is again simply the ending market value divided by the beginning market value minus one, except this time it only covers a period where there was neither cash inflow nor outflow within the position.

Our approach to computing rates of return is consistent with the Global Investment Presentation Standards (GIPS). Mathematically, the interim rate of return (*IRoR*) can be expressed as[9]

$$IRoR = \frac{EMV + Inc - BMV}{BMV}, \textbf{ (Interim rate of return)} \quad (143)$$

where EMV denotes the ending market value, Inc denotes any transfer of consideration due to owning the position, and BMV denotes the beginning market value. Note that equation (1) will be in decimal form and often rates of return are reported in percentage form, hence *IRoR* would be multiplied by 100.

For example, suppose you own 1,000 shares of FRM stock that was trading for \$10 per share on January 1st and it was trading for \$11 per share on March 15th when it paid a \$0.10 dividend. The interim rate of return from January 1st to March 15th is

$$IRoR = \frac{EMV + Inc - BMV}{BMV} = \frac{1{,}000(11) + 1{,}000(0.10) - 1{,}000(10)}{1{,}000(10)} = \frac{11{,}000 + 100 - 10{,}000}{10{,}000} = 0.11$$

or 11 percent.

Time-weighted rate of return

GIPS require the time-weighted rate of return be computed and external cash flows incorporated. The key insight is that the time-weighted rate of return method assumes all interim cash flows are at once reinvested in the existing position. Specifically, GIPS defines time-weighted rate of return as "(a) method of calculating period-by-period returns that negates the effects of external cash flows" and defines external cash flow as "(c)apital (cash or investments) that enters or exits a portfolio." Early in the development of GIPS, the time-weighted approach was explained in the following way:[10]

> *If cash flows occur during the period, they must theoretically be used, in effect, "to buy additional units" of the portfolio at the market price on the day that they are received. Thus the most accurate approach is to calculate the market value of the portfolio on the date of each cash flow, calculate an interim rate of return for the subperiod, and then link the subperiod returns to get the return for the month or quarter.*

The time-weighted rates of return are superior to other methods of computing rates of return because it appropriately handles the timing of cash flows and appropriately handles the reinvestment of these cash flows. The key insight is that anything of value received from an investment is reinvested in that same

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[8]Based, in part, on the *Global Investment Performance Standards*, 2010, CFA Institute.

[9]We assume discrete compounding here as it is the most common approach within the industry. For internal analysis, we also use the continuous compounding approach, or $\ln[(EMV + Inc)/BMV]$.

[10]*Report of the Performance Presentation Standards Implementation Committee*, AIMR, December 1991, p. 28. Note AIMR denotes the Association of Investment Management and Research, the predecessor organization of the CFA Institute. << find updated source >>

investment. Hence, if an owner of a real estate investment receives a two-week time share in Orlando Florida, then it must be assumed that the time share was immediately sold, and the proceeds reinvested in the real estate investment.

There are two ways to illustrate the appropriate way to compute rates of return that are following GIPS, the linking method and the index method. We illustrate both methods using a simple example of a dividend paying stock. Table B1 presents the relevant inputs.

Table B1. Inputs for Computing Time Weighted Rate of Return

Date	Dollar Dividend Per Share	Other Events	Market Price
January 1			$100
March 15	$1		$105
June 15	$1		$110
September 15	$1		$108
October 1		2-for-1 Stock Split	$52 (post split price)
December 15	$0.5		$51
December 31			$50

What is the GIPS-compliant rate of return if the portfolio manager started the year with 100 shares and did not trade this stock? The time-weighted rate of return by the linking method is found by computing the interim rates of return and then linking them together in the same manner as used when computing the geometric average rate of return. The linking method is the method of choice for computational implementation of rates of return calculations. Table B2 illustrates the computation of the time weighted rate of return by the linking method.

Table B2. Linking Method Illustrated for Computing Time Weighted Rate of Return

Date	Interim Period	Interim Rate of Return	Time-Weighted Rate of Return
January 1			
March 15	1	0.0600000 (1)	0.0600000
June 15	2	0.0571486 (2)	0.1205775 (7)
September 15	3	-0.0090909 (3)	0.1103905
October 1	4	-0.0370370 (4)	0.0692649
December 15	5	-0.0096154 (5)	0.0589835
December 31	6	-0.0196078 (6)	0.0382192 (8)

(1) (105 + 1 – 100)/100 = 0.06
(2) (110 + 1 – 105)/105 = 0.0571486
(3) (108 + 1 – 110/110 = –0.090909
(4) (2(52) – 108)/108 = –0.0370370
(5) (51 + 0.5 – 52)/52 = –0.0096154
(6) (50 – 51)/51 = –0.0196078
(7) (1 + 0.06)(1 + 0.0571486) – 1 = 0.1205775
(8) (1 + 0.0589835)(1 – 0.0196078) – 1 = 0.0382192

Thus, the time-weighted rate of return over this year was 3.82 percent. Note that this stock paid $4 in dividends on a $100 stock or a 4 percent dividend yield. Also, the stock started and finished the year at the same pre-split price of $100. Why isn't the rate of return 4 percent? The index method of computing rates of return makes it clear why the rate of return was lower in this case. The index method illustrates the reinvestment of all interim cash flows back into the position. The index method highlights the intuition.

Thus, an index of the number of shares held is constructed and adjusted over the period. Table B3 illustrates the computation of the time weighted rate of return by the index method.

Table B3. Index Method Illustrated for Computing Time Weighted Rate of Return

Date	Dividend and Other Events	Additional Shares Purchased	Number of Shares Owned
January 1			100
March 15	\$1	0.952381 (1)	100.952381 (2)
June 15	\$1	0.917749 (3)	101.870130 (4)
September 15	\$1	0.943242 (5)	102.813372 (6)
October 1	2-for-1 Split		205.626744 (7)
December 15	\$0.5	2.015948 (8)	207.642692 (9)
December 31			207.642692

(1) 1(100)/105 = 0.952381
(2) 100 + 0.952381 = 100.0952381
(3) 1(100.952381)/110 = 0.917749
(4) 100.952381 + 0.917749 = 101.870130
(5) 1(101.870130)/108 = 0.943242
(6) 101.870130 + 0.943242 = 102.813372
(7) 2(102.813372) = 205.626744
(8) 0.5(205.626744) = 2.015948
(9) 205.626744 + 2.015948 = 207.642692

Thus, the rate of return is

$$R_0 = \frac{EMV + Inc - BMV}{BMV} = \frac{207.642692(50) - 100(100)}{100(100)} = 0.038213.$$

The difference between the linking method and the index method is solely rounding error.

Appendix C. Return Attribution Analysis

<< CHANGE NOTATION AND REWRITE >>

For the managed portfolio, the value of the managed portfolio at time t ($\Pi_{M,t}$) can be represented as

$$\Pi_{M,t} = \sum_{j=1}^{n_M} N_{M,j,t} P_{j,t}, \tag{16.2.144}$$

where $N_{M,j,t}$ is the number of shares of security j held in the managed portfolio at time t (for example, shares of stock) and $P_{j,t}$ is the value of one share of security j held in the managed portfolio observed at time t (for example, stock price per share). Let n_M denote the total number of instruments owned in the managed portfolio at point in time t.

Managed security portfolios are evaluated against a benchmark portfolio that is also known as a bogey. Following the notation above with B denoting benchmark or bogey,

$$\Pi_{B,t} = \sum_{j=1}^{n_B} N_{B,j,t} P_{j,t}, \tag{16.2.145}$$

where $N_{B,j,t}$ is the number of shares of security j represented in the benchmark portfolio at time t and $P_{j,t}$ is the value of one share of security j held in the benchmark portfolio observed at time t. Let n_B denote the total number of instruments owned in the benchmark portfolio at point in time t.
The percentage rate of return over period t (length of time unspecified, formally from time t to time $t+1$) of the managed portfolio ($\tilde{R}_{M,t}$) and the benchmark portfolio ($\tilde{R}_{B,t}$) can be expressed as:

$$\tilde{R}_{M,t} = \sum_{j=1}^{n_M} w_{M,j,t} \tilde{R}_{j,t} \text{ and} \tag{16.2.146}$$

$$\tilde{R}_{B,t} = \sum_{j=1}^{n_B} w_{B,j,t} \tilde{R}_{j,t}, \tag{16.2.147}$$

where $w_{M,j,t}$ ($w_{B,j,t}$) denotes the proportion of managed portfolio Π_M (Π_B) invested in security j at point in time t and $\tilde{R}_{j,t}$ denotes the rate of return on security j over period t. Formally,

$$w_{M,j,t} = \frac{N_{M,j,t} P_{M,j,t}}{\sum_{j=1}^{n_M} N_{M,j,t} P_{M,j,t}}, \tag{16.2.148}$$

$$w_{B,j,t} = \frac{N_{B,j,t} P_{B,j,t}}{\sum_{j=1}^{n_B} N_{B,j,t} P_{B,j,t}}, \text{ and} \tag{16.2.149}$$

$$\tilde{R}_{j,t} = \frac{\tilde{P}_{j,t+1}}{P_{j,t}} - 1. \tag{16.2.150}$$

Define the excess return as

$$ER_t = \tilde{R}_{M,t} - \tilde{R}_{B,t} = \sum_{j=1}^{n_M} w_{M,j,t} \tilde{R}_{j,t} - \sum_{j=1}^{n_B} w_{B,j,t} \tilde{R}_{j,t}. \tag{16.2.151}$$

Note that the excess return can be decomposed into the sector allocation decision, the security selection decision, and an interaction term

$$
\begin{aligned}
ER_t &= SAD_t + SSD_t + I_t = \sum_{k=1}^{K}\left(SAW_{M,k,t} - SAW_{B,k,t}\right)\tilde{R}_{B,k,t} \\
&+\sum_{k=1}^{K} SAW_{B,k,t}\left(\tilde{R}_{M,k,t} - \tilde{R}_{B,k,t}\right) + \sum_{k=1}^{K}\left(SAW_{M,k,t} - SAW_{B,k,t}\right)\left(\tilde{R}_{M,k,t} - \tilde{R}_{B,k,t}\right), \\
&= \sum_{k=1}^{K} SAW_{M,k,t}\tilde{R}_{M,k,t} - SAW_{B,k,t}\tilde{R}_{B,k,t}
\end{aligned}
\tag{16.2.152}
$$

where

$$SAW_{M,k,t} = \sum_{j=1}^{n_M} w_{M,j,t} I_{\text{Sector}_k}\left(j\right) \text{ and} \tag{16.2.153}$$

$$SAW_{B,k,t} = \sum_{j=1}^{n_B} w_{B,j,t} I_{\text{Sector}_k}\left(j\right), \tag{16.2.154}$$

where $I_{\text{Sector}_k}\left(j\right)$ denotes an indicator function that equals 1.0 when stock j falls within sector k and zero otherwise. Note, by definition,

$$1.0 = \sum_{k=1}^{K} SAW_{M,k,t} \text{ and} \tag{16.2.155}$$

$$1.0 = \sum_{k=1}^{K} SAW_{B,k,t}, \tag{16.2.156}$$

where K denotes the number of sectors.

The weighted average return for each sector for the managed portfolio and the benchmark portfolio:

$$\tilde{R}_{M,k,t} = \sum_{j=1}^{n_M}\left(\frac{w_{M,j,t}}{SAW_{M,k,t}}\right)\tilde{R}_{j,t} I_{\text{Sector}_k}\left(j\right) \text{ and} \tag{16.2.157}$$

$$\tilde{R}_{B,k,t} = \sum_{j=1}^{n_B}\left(\frac{w_{B,j,t}}{SAW_{B,k,t}}\right)\tilde{R}_{j,t} I_{\text{Sector}_k}\left(j\right). \tag{16.2.158}$$

The excess return allocated to the sector allocation decision can be estimated as

$$SAD = \sum_{k=1}^{K}\left(SAW_{M,k,t} - SAW_{B,k,t}\right)\tilde{R}_{B,k,t}. \tag{16.2.159}$$

The excess return allocated to the security selection decision can be estimated as

$$SSD = \sum_{k=1}^{K} SAW_{B,k,t}\left(\tilde{R}_{M,k,t} - \tilde{R}_{B,k,t}\right). \tag{16.2.160}$$

The excess return allocated to the interaction term can be estimated as

$$I = \sum_{k=1}^{K}\left(SAW_{M,k,t} - SAW_{B,k,t}\right)\left(\tilde{R}_{M,k,t} - \tilde{R}_{B,k,t}\right). \tag{16.2.161}$$

Measures the portion of the excess return attributed to both the sector allocation decision and the security selection decision.

Potential reported return statistics (note time subscript suppressed for clarity)
All of the statistics below can be reported by entire measurement period, subperiods, or ex ante based on given theoretical model.

Return on the *managed* portfolio in total and by sectors:

$$\tilde{R}_M = \sum_{j=1}^{n_M} w_{M,j}\tilde{R}_j = \sum_{k=1}^{K} SA\tilde{R}_{M,k} = \sum_{k=1}^{K} SAW_{M,k}\tilde{R}_{M,k}, \qquad (16.2.162)$$

$$SA\tilde{R}_{M,k} = SAW_{M,k}\tilde{R}_{M,k} \text{ weight-adjusted, sector allocation return,} \qquad (16.2.163)$$

$$SAW_{M,k} = \sum_{j=1}^{n_M} w_{M,j} I_{\mathrm{Sec}tor_k}(j) \text{ sector allocation weight, and} \qquad (16.2.164)$$

$$\tilde{R}_{M,k} = \sum_{j=1}^{n_M}\left(\frac{w_{M,j}}{SAW_{M,k}}\right)\tilde{R}_j I_{\mathrm{Sec}tor_k}(j) \text{ return to sector k.} \qquad (16.2.165)$$

Return on the *benchmark* portfolio in total and by sectors:

$$\tilde{R}_B = \sum_{j=1}^{n_B} w_{B,j}\tilde{R}_j = \sum_{k=1}^{K} SA\tilde{R}_{B,k} = \sum_{k=1}^{K} SAW_{B,k}\tilde{R}_{B,k}, \qquad (16.2.166)$$

$$SA\tilde{R}_{B,k} = SAW_{B,k}\tilde{R}_{B,k} \text{ weight-adjusted, sector allocation return,} \qquad (16.2.167)$$

$$SAW_{B,k} = \sum_{j=1}^{n_B} w_{B,j} I_{\mathrm{Sec}tor_k}(j) \text{ sector allocation weight, and} \qquad (16.2.168)$$

$$\tilde{R}_{B,k} = \sum_{j=1}^{n_B}\left(\frac{w_{B,j}}{SAW_{B,k}}\right)\tilde{R}_j I_{\mathrm{Sec}tor_k}(j) \text{ return to sector k.} \qquad (16.2.169)$$

Excess return in total and by decisions:

$$ER = \tilde{R}_M - \tilde{R}_B = \sum_{j=1}^{n_M} w_{M,j}\tilde{R}_j - \sum_{j=1}^{n_B} w_{B,j}\tilde{R}_j = SAD + SSD + I \text{ excess return.} \qquad (16.2.170)$$

Sector allocation decision in total and by sector:

$$SAD = \sum_{k=1}^{K} SAD_k = \sum_{k=1}^{K}\left(SAW_{M,k} - SAW_{B,k}\right)\tilde{R}_{B,k} \text{ total sector allocation decision,} \qquad (16.2.171)$$

$$SAD_k = \left(SAW_{M,k} - SAW_{B,k}\right)\tilde{R}_{B,k} \text{ sector allocation decision attributable to sector k, and} \qquad (16.2.172)$$

$$SAW_{M,k} - SAW_{B,k} \text{ excess weight allocated to sector k.} \qquad (16.2.173)$$

Security selection decision in total and by sector:

$$SSD = \sum_{k=1}^{K} SSD_k = \sum_{k=1}^{K} SAW_{B,k}\left(\tilde{R}_{M,k} - \tilde{R}_{B,k}\right) \text{ total security selection decision,} \qquad (16.2.174)$$

$$SSD_k = SAW_{B,k}\left(\tilde{R}_{M,k} - \tilde{R}_{B,k}\right) \text{ security selection decision attributable to sector k, and} \qquad (16.2.175)$$

$$\tilde{R}_{M,k} - \tilde{R}_{B,k} \text{ excess return allocated to sector k.} \qquad (16.2.176)$$

Interaction in total and by sector:

$$I = \sum_{k=1}^{K} I_k = \sum_{k=1}^{K} \left(SAW_{M,k} - SAW_{B,k}\right)\left(\tilde{R}_{M,k} - \tilde{R}_{B,k}\right) \text{ total interaction,} \quad (16.2.177)$$

$$I_k = \left(SAW_{M,k} - SAW_{B,k}\right)\left(\tilde{R}_{M,k} - \tilde{R}_{B,k}\right) \text{ interaction attributable to sector k,} \quad (16.2.178)$$

$$SAD_k = \left(SAW_{M,k} - SAW_{B,k}\right)\tilde{R}_{B,k} \text{ sector allocation decision attributable to sector k,} \quad (16.2.179)$$

$$SSD_k = SAW_{B,k}\left(\tilde{R}_{M,k} - \tilde{R}_{B,k}\right) \text{ security selection decision attributable to sector k,} \quad (16.2.180)$$

$$SAW_{M,k}\left(\tilde{R}_{M,k} - \tilde{R}_{B,k}\right) \text{ managed portfolio weight-adjusted excess return attributable to sector k, and}$$

(16.2.181)

$$\left(SAW_{M,k} - SAW_{B,k}\right)\tilde{R}_{M,k} \text{ managed return adjusted by excess weight allocated to sector k.} (16.2.182)$$

Subperiod aggregation:

$$ER_t = \tilde{R}_{M,t} - \tilde{R}_{B,t} = \sum_{j=1}^{n_M} w_{M,j,t}\tilde{R}_{j,t} - \sum_{j=1}^{n_B} w_{B,j,t}\tilde{R}_{j,t}.$$

Example << to be developed >>

<< see spreadsheet >>

Performance Attribution							
Asset Allocation							
	Benchmark		**Managed**				
	Weights	**Return**	**Weights**	**Return**	**SAD**	**SSD**	**Interaction**
Stocks	60.00%	6.00%	70.00%	7.00%	0.6000%	0.6000%	0.1000%
Bonds	40.00%	3.00%	25.00%	2.50%	-0.4500%	-0.2000%	0.0750%
Cash	0.00%	1.00%	5.00%	1.20%	0.0500%	0.0000%	0.0100%
Total Weights	100.00%		100.00%				
Return		4.8000%		5.5850%	0.2000%	0.4000%	0.1850%
Excess Return				0.7850%			0.7850%
Sector Allocation Decision	SAD (Stocks)=	=($D5-$B5)*$C5					
Security Selection Decision	SSD (Stocks)=	=$B5*($E5-$C5)					
Interaction	I (Stocks)=	=($D5-$B5)*($E5-$C5)					
	SAW(B)	**SAW(M)**	**EW**				
Stocks	60.00%	70.00%	10.00%				
Bonds	40.00%	25.00%	-15.00%				
Cash	0.00%	5.00%	5.00%				
Total Weights	100.00%	100.00%	0.00%				
Notional Portfolios	**Return**						
1) W(B)*R(B)	4.80%	Benchmark Portfolio					
2) W(B)*R(M)	5.20%	Benchmark weights, Managed returns					
3) W(M)*R(B)	5.00%	Managed returns, Benchmark weights					
4) W(M)*R(M)	5.59%	Managed Portfolio					
3) - 1): SAD	0.20%	Sector Allocation Decision					
2) - 1): SSD	0.40%	Security Selectiion Decision					

Assumptions

- Single measurement period (issues are much more complicated in multi-period setting)
- No portfolio turnover during single period or approximations made based on weighted average of holding period

Appendix D. Risk Attribution Analysis

<< CHANGE NOTATION AND REWRITE >>

The statistics are not additive; hence, subperiod results do not aggregate to entire measurement period. Typically, risk is not additive across time, however percentage marginal contribution to risk is additive across sectors or instruments.

Basic risk setup

The total risk of the portfolio can be measured as the variance of the rate of return on the portfolio or

$$\sigma_{\Pi}^2 \equiv E\left\{\left[\tilde{R}_{\Pi} - E\left(\tilde{R}_{\Pi}\right)\right]^2\right\} = Cov\left(\tilde{R}_{\Pi}, \tilde{R}_{\Pi}\right) = Cov\left(\sum_{j=1}^{n} w_j \tilde{R}_j, \tilde{R}_{\Pi}\right). \tag{16.2.183}$$

Variance is not the only risk measure. Other candidates include downside risk, value-at-risk, conditional value-at-risk, beta, and so forth.

Recall covariance [$Cov(\ )$] is the expectation of the product of the deviations from the mean of which variance is a special case. The last equality is a direct substitution from the definition of return. From the properties of covariance, we have:

$$\sigma_{\Pi}^2 = Cov\left(\sum_{j=1}^{n} w_j \tilde{R}_j, \tilde{R}_{\Pi}\right) = \sum_{j=1}^{n} w_j Cov\left(\tilde{R}_j, \tilde{R}_{\Pi}\right). \tag{16.2.184}$$

Thus the marginal contribution to risk of any given security j (denote $MCTR_j$) within a portfolio is:

$$MCTR_j = w_j Cov\left(\tilde{R}_j, \tilde{R}_{\Pi}\right). \tag{16.2.185}$$

Clearly the sum of $MCTR_j$ is equal to the portfolio variance. Also note that if the covariance is negative, the contribution to risk is negative (assuming a long position or $w_j > 0$). Dividing both sides by the portfolio variance we can compute the percentage marginal contribution to risk as

$$1 = \frac{\sum_{j=1}^{n} MCTR_j}{\sigma_{\Pi}^2} = \frac{\sum_{j=1}^{n} w_j Cov\left(\tilde{R}_j, \tilde{R}_{\Pi}\right)}{\sigma_{\Pi}^2} = \sum_{j=1}^{n} w_j \beta_{j,\Pi} = \sum_{j=1}^{n} \%MCTR_j . \tag{16.2.186}$$

Therefore,

$$\%MCTR_j = w_j \beta_{j,\Pi}, \tag{16.2.187}$$

where

$$\beta_{j,\Pi} = \frac{Cov\left(\tilde{R}_j, \tilde{R}_{\Pi}\right)}{\sigma_{\Pi}^2}. \tag{16.2.188}$$

Beta could be estimated historically as the slope of an ordinary least squares regression. The total risk as measured by variance can be decomposed into the risk related to each security.

Application to Brinson attribution analysis

Decomposition of tracking error risk (excess return risk): ($\sigma_{ER}^2 > 0$)

$$1=\frac{Cov\left(\tilde{R}_M-\tilde{R}_B,E\tilde{R}\right)}{\sigma_{ER}^2}=\frac{Cov\left(\tilde{R}_M,E\tilde{R}\right)}{\sigma_{ER}^2}-\frac{Cov\left(\tilde{R}_B,E\tilde{R}\right)}{\sigma_{ER}^2}=\beta_{M,ER}-\beta_{B,ER}. \tag{16.2.189}$$

Decomposition of sector allocation decision, security selection decision and interaction:

$$\begin{aligned}1&=\frac{Cov\left(S\tilde{A}D+S\tilde{S}D+\tilde{I},E\tilde{R}\right)}{\sigma_{ER}^2}=\frac{Cov\left(S\tilde{A}D,E\tilde{R}\right)}{\sigma_{ER}^2}+\frac{Cov\left(S\tilde{S}D,E\tilde{R}\right)}{\sigma_{ER}^2}+\frac{Cov\left(\tilde{I},E\tilde{R}\right)}{\sigma_{ER}^2},\\&=\beta_{SAD,ER}+\beta_{SSD,ER}+\beta_{I,ER}\end{aligned} \tag{16.2.190}$$

where

$$\beta_{SAD,ER}=\frac{Cov\left(S\tilde{A}D,E\tilde{R}\right)}{\sigma_{ER}^2}=\sum_{k=1}^{K}\left(SAW_{M,k}-SAW_{B,k}\right)\beta_{B,k,ER}, \tag{16.2.191}$$

$$\beta_{SSD,ER}=\frac{Cov\left(S\tilde{S}D,E\tilde{R}\right)}{\sigma_{ER}^2}=\sum_{k=1}^{K}SAW_{B,k}\left(\beta_{M,k,ER}-\beta_{B,k,ER}\right), \tag{16.2.192}$$

$$\beta_{I,ER}=\frac{Cov\left(\tilde{I},E\tilde{R}\right)}{\sigma_{ER}^2}=\sum_{k=1}^{K}\left(SAW_{M,k}-SAW_{B,k}\right)\left(\beta_{M,k,ER}-\beta_{B,k,ER}\right), \tag{16.2.193}$$

$$\beta_{M,k,ER}=\frac{\operatorname{cov}\left(\tilde{R}_{M,k},E\tilde{R}\right)}{\sigma_{ER}^2}, \text{ and} \tag{16.2.194}$$

$$\beta_{B,k,ER}=\frac{\operatorname{cov}\left(\tilde{R}_{B,k},E\tilde{R}\right)}{\sigma_{ER}^2}. \tag{16.2.195}$$

Note

$$\beta_{SAD,ER}+\beta_{SSD,ER}+\beta_{I,ER}=\beta_{M,ER}-\beta_{B,ER}=1. \tag{16.2.196}$$

Potential reported risk statistics (note time subscript suppressed for clarity)
Risk of the *managed* portfolio in total and by sectors:

$$\sigma^2\left(\tilde{R}_M\right)=\sum_{j=1}^{n_M}w_{M,j}\operatorname{cov}\left(\tilde{R}_j,\tilde{R}_M\right)=\sum_{k=1}^{K}\operatorname{cov}\left(SA\tilde{R}_{M,k},\tilde{R}_M\right)=\sum_{k=1}^{K}SAW_{M,k}\operatorname{cov}\left(\tilde{R}_{M,k},\tilde{R}_M\right). \tag{16.2.197}$$

Percentage risk of the *managed* portfolio in total and by sectors:

$$1=\sum_{j=1}^{n_M}w_{M,j}\beta_{j,M}=\sum_{k=1}^{K}\beta_{SAR_{M,k},M}=\sum_{k=1}^{K}SAW_{M,k}\beta_{R_{M,k},M}, \tag{16.2.198}$$

$$\beta_{SAR_{M,k},M}=\frac{\operatorname{cov}\left(SA\tilde{R}_{M,k},\tilde{R}_M\right)}{\sigma^2\left(\tilde{R}_M\right)} \text{ weight-adjusted, sector allocation percentage risk,} \tag{16.2.199}$$

$$SAW_{M,k}=\sum_{j=1}^{n_M}w_{M,j}I_{\text{Sector}_k}\left(j\right) \text{ sector allocation weight, and} \tag{16.2.200}$$

$$\beta_{R_{M,k},M}=\frac{\operatorname{cov}\left(\tilde{R}_{M,k},\tilde{R}_M\right)}{\sigma^2\left(\tilde{R}_M\right)} \text{ percentage risk for sector k.} \tag{16.2.201}$$

Risk of the *benchmark* portfolio in total and by sectors:

$$\sigma^2\left(\tilde{R}_B\right)=\sum_{j=1}^{n_B} w_{B,j}\operatorname{cov}\left(\tilde{R}_j,\tilde{R}_B\right)=\sum_{k=1}^{K}\operatorname{cov}\left(SA\tilde{R}_{B,k},\tilde{R}_B\right)=\sum_{k=1}^{K} SAW_{B,k}\operatorname{cov}\left(\tilde{R}_{B,k},\tilde{R}_B\right). \quad (16.2.202)$$

Percentage risk of the *benchmark* portfolio in total and by sectors:

$$1=\sum_{j=1}^{n_B} w_{B,j}\beta_{j,B}=\sum_{k=1}^{K}\beta_{SAR_{B,k},B}=\sum_{k=1}^{K} SAW_{B,k}\beta_{R_{B,k},B},$$

$$\beta_{SAR_{B,k},B}=\frac{\operatorname{cov}\left(SA\tilde{R}_{B,k},\tilde{R}_B\right)}{\sigma^2\left(\tilde{R}_B\right)} \text{ weight-adjusted, sector allocation percentage risk,} \quad (16.2.203)$$

$$SAW_{B,k}=\sum_{j=1}^{n_B} w_{B,j} I_{\text{Sector}_k}\left(j\right) \text{ sector allocation weight, and} \quad (16.2.204)$$

$$\beta_{SAR_{B,k},B}=\frac{\operatorname{cov}\left(\tilde{R}_{B,k},\tilde{R}_B\right)}{\sigma^2\left(\tilde{R}_B\right)} \text{ percentage risk for sector k.} \quad (16.2.205)$$

Risk of the excess return in total and by sectors:

$$\begin{aligned}\sigma^2\left(ER\right)&=\operatorname{cov}\left(\tilde{R}_M-\tilde{R}_B,ER\right)=\sum_{j=1}^{n_M} w_{M,j}\operatorname{cov}\left(\tilde{R}_j,ER\right)-\sum_{j=1}^{n_B} w_{B,j}\operatorname{cov}\left(\tilde{R}_j,ER\right).\\&=\operatorname{cov}\left(SAD+SSD+I,ER\right)=\operatorname{cov}\left(SAD,ER\right)+\operatorname{cov}\left(SSD,ER\right)+\operatorname{cov}\left(I,ER\right)\end{aligned} \quad (16.2.206)$$

Percentage risk of the excess return by portfolio and by decisions:

$$\begin{aligned}1&=\beta_{M,ER}-\beta_{B,ER}=\sum_{j=1}^{n_M} w_{M,j}\beta_{j,ER}-\sum_{j=1}^{n_B} w_{B,j}\beta_{j,ER}=\sum_{j=1}^{n_M\cup n_B}\left(w_{M,j}-w_{B,j}\right)\beta_{j,ER}.\\&=\beta_{SAD,ER}+\beta_{SSD,ER}+\beta_{I,ER}\end{aligned} \quad (16.2.207)$$

Sector allocation decision percentage risk in total and by sector:

$$\beta_{SAD,ER}=\sum_{k=1}^{K}\beta_{SAD,k,ER}=\sum_{k=1}^{K}\left(SAW_{M,k}-SAW_{B,k}\right)\beta_{B,k,ER} \text{ total SAD percentage risk,} \quad (16.2.208)$$

$$\beta_{SAD,k,ER}=\left(SAW_{M,k}-SAW_{B,k}\right)\beta_{B,k,ER} \text{ SAD attributable to sector k percentage risk,} \quad (16.2.209)$$

$$SAW_{M,k}-SAW_{B,k} \text{ excess weight allocated to sector k.} \quad (16.2.210)$$

Security selection decision percentage risk in total and by sector:

$$\beta_{SSD,ER}=\sum_{k=1}^{K}\beta_{SSD,k,ER}=\sum_{k=1}^{K} SAW_{B,k}\left(\beta_{M,k,ER}-\beta_{B,k,ER}\right) \text{ total SSD percentage risk,} \quad (16.2.211)$$

$$\beta_{SSD,k,ER}=SAW_{B,k}\left(\beta_{M,k,ER}-\beta_{B,k,ER}\right) \text{ SSD percentage risk attributable to sector k, and} \quad (16.2.212)$$

$$\beta_{M,k,ER}-\beta_{B,k,ER} \text{ excess percentage risk allocated to sector k.} \quad (16.2.213)$$

Interaction percentage risk (IPR) in total and by sector:

$$\beta_{I,ER}=\sum_{k=1}^{K}\beta_{I,k,ER}=\sum_{k=1}^{K}\left(SAW_{M,k}-SAW_{B,k}\right)\left(\beta_{M,k,ER}-\beta_{B,k,ER}\right) \text{ total IPR,} \quad (16.2.214)$$

$$\beta_{I,k,ER}=\left(SAW_{M,k}-SAW_{B,k}\right)\left(\beta_{M,k,ER}-\beta_{B,k,ER}\right) \text{ IPR attributable to sector k,} \quad (16.2.215)$$

$$\beta_{SAD,k,ER}=\left(SAW_{M,k}-SAW_{B,k}\right)\beta_{B,k,ER} \text{ SAD attributable to sector k percentage risk,} \quad (16.2.216)$$

$\beta_{SSD,k,ER} = SAW_{B,k}\left(\beta_{M,k,ER} - \beta_{B,k,ER}\right)$ SSD percentage risk attributable to sector k, (16.2.217)

$SAW_{M,k}\left(\beta_{M,k,ER} - \beta_{B,k,ER}\right)$ managed portfolio weight-adjusted excess % risk attributable to sector k, and (16.2.218)

$\left(SAW_{M,k} - SAW_{B,k}\right)\beta_{M,k,ER}$ managed percentage risk adjusted by excess weight allocated to sector k. (16.2.219)